

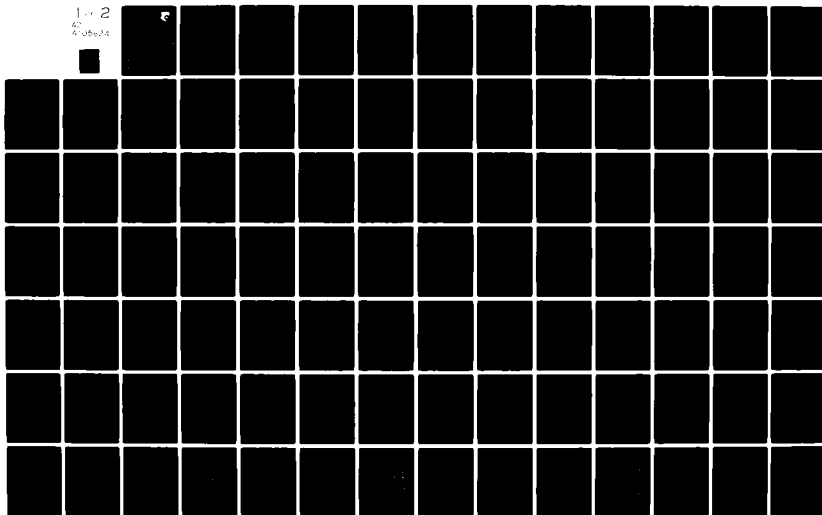
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DETERMINATION OF COMPLEX CHARACTERISTIC
VALUES AND VECTORS FROM SINUSOIDAL
EXCITATIONS AT NEAR RESONANCE FREQUENCIES

C. L. Keller

Applied Mathematics Group
Analysis and Optimization Branch

June 1981

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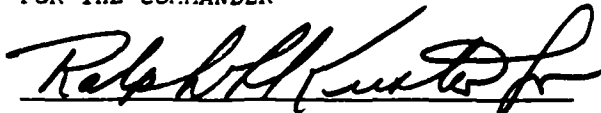


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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this report an iterative procedure for determining complex characteristic values and vectors from measured values of the steady state response to sinusoidal excitations is described. An extension of the procedure which enables it to cope with the case of two close characteristic values is presented also. Results of numerical experiments performed to investigate these procedures are given. This report contains also extensive background material and an examination of the methods of Ibrahim, of Wittmeyer and of Link and Vullan.			

JOB

FOREWORD

This report describes work performed in the Applied Mathematics Group of the Analysis and Optimization Branch, Structures and Dynamics Division of the Flight Dynamics Laboratory (AFWAL/FIBRD) under Project 2304N1 Computational Aspects of Fluid and Structural Mechanics, Work Unit 2304N102. This is an interim report on work carried out between August 1978 and June 1980. The author, C. L. Keller, submitted the report in October 1980.

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SECTION I

INTRODUCTION

In the last 10 years a number of procedures have been proposed for determining vibration parameters of a structure from its measured response to known excitations. The reason for this activity was due to shortcomings and dissatisfaction with the classical phase resonance or tune and dwell procedure.

In the past, the phase resonance method was characterized as requiring a rather lengthy set up time. Practical limits on the number and positioning of the shakers result in limitations on the accuracy of the method. The desired information is obtained directly from the test. However, obtaining satisfactory test results was a slow process requiring highly skilled operators. Consequently, the test structure was not available for other programmed activities for a lengthy period, usually, at a crucial time.

Accordingly, other methods were sought for which the test set up would be simpler, require fewer excitation points, and be less dependent upon operator skill. The desired information need not be given directly by the test but would be obtained at a later date from the test data after the structure was released.

For those methods for which test results are not known until some time after the structure is released there is a danger that, due to an undetected error in the test set up, the test data is worthless. Frequently, in such cases it is impossible to reschedule the test until after such a time that the main reason for the test is past.

In Section II of this report a procedure is described for computing the complex characteristic values and vectors from the steady state response to sinusoidal excitations at a single point. In Section III we describe a modification of the procedure so that it can handle the

case of two close characteristic values. In a previous report, AFFDL-TR-78-59 (Reference 1), we have shown how to modify the procedure to handle characteristic values of multiplicity greater than 1.

We believe the procedure described in Sections II and III has most of the advantages of the phase resonance method with most of the shortcomings eliminated. Since it depends upon excitations at a single point the test set up is simpler. Also, since data only in the vicinity of resonance is used, only the frequency needs to be adjusted. The frequency adjustments do not have to be too precise, consequently, the degree of operator skill is of less importance.

The computations are simple. We believe the procedure can be automated so that the final results are obtained practically on line. Accordingly, errors in test set up, if any, should be detected early in the allotted time for the tests, corrected and the test rerun. Moreover, the nonlinear damping and stiffness properties can be calculated as functions of the excitation amplitudes.

The procedure described in Sections II and III is based upon excitations at a single point. However, it should be clear from the discussion of these sections that the procedure is not limited to single point excitation. For multipoint excitation, the quantity $v_k^T r$, Equation 2, no longer isolates a single component of the vector v_k but instead, $v_k^T r$ is some linear combination of the components of v_k . Thus for multipoint excitation one needs to perform sufficiently many additional independent experiments to determine at least one (and consequently all) of the components of v_k appearing in the linear combination $v_k^T r$. It is clear also that this is accomplished by varying components of the vector r , $v_k^T r$ that is, by varying the amplitude at the set of multiple excitation points in such a way that the total energy input is essentially constant.

Using the procedure of Sections II and III modified for multiple point excitation does not affect the test set up appreciably. Rather, it amounts to some additional experiments and computations which are essentially of the same character as for single point excitation.

In Section IV we describe a Newton procedure for determining the characteristic values and make some additional remarks concerning the iteration process of Section II. In Section V we describe the numerical experiments which we perform to test the procedure of Sections II and III and give the results of these numerical experiments.

In Appendices A-D we give mathematical background for the method of Section II and for some of the other methods examined. Appendix A is mainly results, from the theory of linear systems of differential equations with constant coefficients, which are helpful for the identification problem. Appendix B develops, in considerable detail, Prony's method and contrasts Prony's method with the method of Ibrahim (References 2-5) for determining the parameters in a sum of exponential functions. Appendix C considers some aspects of the matrix eigenvalue-eigenvector problem and the use thereof in Wittmeyer's method (References 6 and 7). In Appendix D we discuss the problem of solving systems of linear equations which may be over, even, or under determined and the method Link and Vollen (Reference 8).

The methods for determining vibration parameters can be classified in various ways depending upon which features are emphasized. One rather evident category consists of those methods which determine mode shapes and frequencies. That is, numbers λ and vectors u which satisfy the conditions, either

$$[\lambda^2 M + \lambda C + K]u = 0$$

or

$$[\lambda^2 M + K]u = 0$$

A second category consists of those methods which determine the mass, damping and stiffness matrices M , C and K directly. The second category appears to consist of variants of the Link and Vollen method and the "ill-conditioned" method described in Appendix D. Thus most methods belong to the first category.

Many methods use the frequency response function (Equation A38) in some way or other. The frequency response function can be written in different ways depending on how the characteristic vectors are normalized and which vibration parameters are displayed.

The method discussed in References 9-13 computes (a column of) the frequency response function matrix from the response to excitation of impulse or random type. The procedure uses fast Fourier analysis methods (Equation A28) and is highly automated. In this procedure the frequency response function is obtained numerically, that is, as a set of number pairs. The second entry of this number pair is the "value" of the frequency response function and the first entry is the frequency ω to which the second entry corresponds. Lastly, the parameters in the frequency response function are determined, usually, according to the least squares criteria. Thus advances in electronics, in sensing and recording equipment and minicomputers, have made a complicated procedure economically practical.

Hence the fast Fourier analysis procedure discussed in References 9-13 uses broad band excitation to produce a transient response. From the Fourier analysis of this experimentally obtained data one obtains the frequency response function (numerically). Finally, from the "fit" of the expression for the frequency response function to the experimentally determined values of the frequency response function one obtains the complex mode shapes and frequencies.

The method of Section II is based upon the frequency response function matrix and its properties also. The complex amplitude y of the steady state response $y \exp(i\omega t)$ to the harmonic excitation $r \exp(i\omega t)$ is, essentially, the value of the frequency response function corresponding to the frequency ω . From knowledge of the nature of the frequency response function we choose a set of frequencies ω and obtain the values of the frequency response function corresponding to the set of frequencies ω . These values of the frequency response function are obtained experimentally as the complex amplitude y of the steady state

response to the harmonic excitation $r \exp(i\omega t)$. The set of values of the frequency response function, so obtained, are used to set up a system of nonlinear equations for the parameters (coefficients) of the frequency response function. The complex mode shapes and frequencies are obtained as the solution to this nonlinear system of equations.

The method of Wittmeyer (References 6 and 7) is yet another method based upon the frequency response function and its properties. This procedure is a method for solving a matrix eigenvector-eigenvalue problem appropriately modified to be applicable to the ground vibration test problem. Matrix eigenvector-eigenvalue methods are iterative procedures, generally. The experimental data is obtained from multipoint sinusoidal excitation. One alternates between experiment and computation. The current approximation to the eigenvector (being determined) is obtained from experiment. If this approximation is not satisfactory an improved excitation vector is computed from the experimental data and the experiment repeated.

The method of Link and Vollen (Reference 8) determines the damping and stiffness matrices directly from steady state response to sinusoidal excitation data. (The mass matrix M usually is assumed known.) This procedure uses the same kind of data as the method of this report. The damping and stiffness matrices C and K are obtained as the solution to a system of linear equations.

Using the pseudo or generalized inverse, if necessary, the linear system of equations is solvable for the matrices C and K regardless of whether the system of equations for C and K is over, even or under determined. Of course, if the system is under determined then the matrices C and K are even less well known.

An adequate number of linearly independent conditions are needed for a strict solution for the matrices C and K . Equation A29 shows the relation of the matrices M , C and K to the frequency response function. Equation A29 also shows that linearly independent conditions are obtained at, or near to resonance frequencies.

Ibrahim's method (References 2-5) determines the complex eigenvectors and frequencies from the free response of the system. Two matrices are generated from the data obtained from the free response sampled at equal time increments. The complex mode shapes and frequencies are obtained from the eigenvalue and eigenvector solution to the generalized eigenvalue-eigenvector problem associated with these two matrices.

The method produces an equivalent model for the structure. That is, it produces a model which has the same eigenvectors as the structure. However, since no particular normalization is required or enforced, the mass, stiffness and damping matrices cannot be determined by the method; at least, not without some additional experiments and computation.

The phase resonance method has also made use of the development in computers and visual display techniques to bring about improvements in the method. The indicator function, basically, is the sum of the magnitudes of all the in phase amplitudes at all the measurement points. With the on line computation and display of the indicator function an operator can see immediately the effect of a change in amplitude at an excitation point or the effect of a change in the excitation frequency. The phase resonance method is still widely used in ground vibration tests.

SECTION II

DETERMINATION OF THE CHARACTERISTIC VALUES AND VECTORS

In this section a method is described for determining the parameters in an expression for the frequency response function matrix from the steady state response to sinusoidal excitations. The quantities determined are the characteristic values λ_k $k=1, \dots, m$, the corresponding characteristic vectors u_k and the corresponding characteristic vectors v_k of the transposed system. (The notation, conventions and results of Appendix A are assumed as being familiar throughout this report.)

The method described here is essentially the same as in Reference 1. However, in Reference 1 vectors were treated as single entities. That is, very little attention was paid to the individual components except for two instances when the fact was used that one could assume a special value or special values for a component or several components. Here, in order to display certain features with clarity, the process is described in terms of a scalar component.

The frequency response function parameters will be determined by an iterative process. We do not have a formal proof for the convergence of this process. However, we are able to make some observations which lead one to believe that the process is convergent. Additionally, our numerical experience with the iteration process, although limited, has not indicated any difficulties, thus providing evidence that the process is convergent under conditions sufficiently general for our purposes.

Usually, in determining the characteristic values and vectors, one assumes that the characteristic values are well-separated. In this section we make this assumption also. In Reference 1 we showed that the method described here could be modified to handle characteristic values of multiplicity greater than 1. In Section III of this report, we will show how to extend the procedure of this section so that it can handle the case of two close characteristic values also.

The procedure described in References 9-13 analyzes the transient structural response to broadband excitation and uses the relation between the frequency response (or transfer) function matrix and the transforms of the excitation and response as expressed by Equation A28. The procedure to be described in this section uses instead the relation between the frequency response function matrix, the steady state response to an harmonic excitation and the harmonic excitation.

It is assumed that after discretization the equations of motion of the structure are adequately represented by a system of equations of the form

$$M\ddot{x} + C\dot{x} + Kx = f \quad (1)$$

Here M , C and K denote the mass, damping and stiffness matrices respectively. M , C and K are taken as square matrices of order m . In addition, these matrices are usually assumed to be symmetric. The components of the vector x usually are displacements at stations on the structure.

The steady state response of a system of equations of the form of Equation 1 to an harmonic excitation $f(t)=r \exp(i\omega t)$ is $x(t)=y \exp(i\omega t)$. Here r and y denote constant vectors. The frequency response function matrix and the vectors r and y satisfy the condition

$$y = \sum_{k=1}^n \frac{u_k v_k^T r}{\omega - \lambda_k} \quad (2)$$

(Equation A38). The vector y is determined experimentally in some way from the steady state response to the excitation $r \sin \omega t$. One possible way is from the system of equations (Equation A37). The vector r , of course, is chosen, usually with only one component differing from zero. This is the component corresponding to the station on the structure at which the excitation is applied.

The frequency response function matrix may be written in various ways. The particular form is a consequence of the normality conditions imposed

on and satisfied by the characteristic vectors u_k and v_k , $k=1, \dots, m$. For characteristic vectors which satisfy the normality condition expressed by Equation A13, the frequency response function matrix takes the form given in Equation 2. It should be clear that one does not need to know the vectors u_k and v_k normalized so that they satisfy the condition (Equation A13). Rather u_k and v_k are determined so that Equation 2 is satisfied and when so determined, then the normality conditions (Equation A13) are satisfied also.

This enforcement of the normality conditions enables one to determine the matrices M , C , and K after the quantities λ_k , u_k and v_k are determined. Thus after λ_k , u_k and v_k are determined so that Equation 2 is satisfied for the steady state response to any harmonic excitation, the relations expressed by Equations A22, A24 and A25 hold. It should not be inferred that these relations will be used directly in computing M , C and K . Rather some relations derived from Equations A22, A24 and A25 or the procedure given in References 12 and 1 may be more efficient for computing these matrices.

From Equation 2 it is clear that for a vector r with only the p th component different from zero, the vector y is this p th component times the p th column of the frequency response function matrix. It follows then that the j th component of the vector y is a complex valued function $w_j(\omega)$ of the form

$$w_j(\omega) = \sum_{k=1}^m b_{jk}/(i\omega - \lambda_k) + \sum_{k=1}^m \bar{b}_{jk}/(i\omega - \bar{\lambda}_k) \quad (3)$$

If the b_{jk} and λ_k are not known, then the expression for the function $w_j(\omega)$ contains $n=2m$ unknowns. To determine the b_{jk} and λ_k , n independent conditions are needed. These n conditions are provided by the values of $w_j(\omega)$ corresponding to n distinct values of ω . We will show that if we know $w_j(\omega)$ for n judiciously chosen values $\omega = \omega_p$, $p=1, \dots, n$ the resulting system of nonlinear equations is readily solved for the λ_k and b_{jk} by an iteration process.

In Figure 1 we have plotted in a complex plane values of a function $w(\omega)$ of the form expressed by Equation 3 for values of ω ranging from 0.5 to 12 at increments of 0.05. Figures 2 and 3 are plots of values of the magnitude and argument respectively of the same function for the same range of values of ω . These figures display the features which characterize functions $w_j(\omega)$ of the form expressed by Equation 3. The appearance of Figure 3 is determined by the range of values selected for the argument of a complex number.

From Figure 2 it is observed that the local maximum of $|w(\omega)|$ occurs at some value of ω close to $\text{Im}[\lambda_k]$ for $k=1, \dots, m$. From Equation A36 it follows that $|w_j(\omega)|$ is the amplitude of the steady state response to $r \sin \omega t$. Also, in Figure 1, if we would label the plotted points with the value of ω to which they correspond - we would observe that $w(\omega)$ changes most rapidly when ω is close to $\text{Im}[\lambda_k]$. That is, in Figure 1 the values of $w(\omega)$ for which ω is close to $\text{Im}[\lambda_k]$ are those points which are connected by rather large straight line segments.

From the above observations it is clear that we may suppose that n values of ω and the corresponding vector y are known in pairs, say ω_p and ω_{m+p} , with corresponding vectors y_p and y_{m+p} , for $p=1, \dots, m$. The values of ω_p satisfy the conditions

$$\omega_p < \text{Im}[\lambda_p] < \omega_{m+p} \quad (4)$$

And the difference $\omega_{m+p} - \omega_p$ is small relative to the differences $\omega_{p+1} - \omega_{m+p}$ and $\omega_p - \omega_{m+p-1}$. Hence the value of the complex function $w_j(\omega)$, which represents the j th component of the vector $y(\omega)$, is known for n values of ω .

It was noted above that the system of equations obtained from Equation 3 when $w(\omega)$ is known for n values of ω is nonlinear. We will now describe an iterative process for determining values of λ_k and b_{jk} which satisfy the system. Set

$$\hat{w}_{jp}(\omega) = \sum_{\substack{k=1 \\ k \neq p}}^m (b_{jk}/(i\omega - \lambda_k) + \bar{b}_{jk}/(i\omega - \bar{\lambda}_k)) + \bar{b}_{jp}/(i\omega - \bar{\lambda}_p) \quad (5)$$

Then

$$b_{jp}/(i\omega - \lambda_p) = w_j(\omega) - \hat{w}_{jp}(\omega) \quad (6)$$

Solving Equation 6 for b_{jp} we obtain therefrom the equality

$$(i\omega_p - \lambda_p)[w_j(\omega_p) - \hat{w}_{jp}(\omega_p)] = (i\omega_{m+p} - \lambda_p)[w_j(\omega_{m+p}) - \hat{w}_{jp}(\omega_{m+p})] \quad (7)$$

These equations, Equation 7 solved for λ_p and Equation 6 solved for b_{jp} are the basic equations of the iteration process.

We show next how to determine starting values or initial approximations for λ_p and b_{jp} . The principal observation is that if ω_p and $\omega_m + p$ are close to $\text{Im}[\lambda_p]$ then, from Equation 3

$$w_j(\omega_p) = b_{jp}/(i\omega_p - \lambda_p) \text{ approximately}$$

and

$$w_j(\omega_{m+p}) = b_{jp}/(i\omega_{m+p} - \lambda_p) \text{ approximately.}$$

Hence

$$w_j(\omega_p)/w_j(\omega_{m+p}) = (i\omega_{m+p} - \lambda_p)/(i\omega_p - \lambda_p)$$

which is readily solved for λ_p , see Equation 10 below. Alternatively, from Equation 7 we obtain

$$\begin{aligned} \lambda_p [w_j(\omega_{m+p}) - w_j(\omega_p) - (\hat{w}_{jp}(\omega_{m+p}) - \hat{w}_{jp}(\omega_p))] = \\ i\omega_p [w_j(\omega_{m+p}) - w_j(\omega_p) - (\hat{w}_{jp}(\omega_{m+p}) - \hat{w}_{jp}(\omega_p))] \\ + i(\omega_{m+p} - \omega_p) [w_j(\omega_{m+p}) - \hat{w}_{jp}(\omega_{m+p})] \end{aligned} \quad (8)$$

From the graph of $\hat{w}_{jp}(\omega)$, Figures 4 and 5, it is evident that $\hat{w}_{jp}(\omega_{m+p}) - \hat{w}_{jp}(\omega_p)$ is small, especially if $\omega_m + p - \omega_p$ is small. Hence neglecting the difference $\hat{w}_{jp}(\omega_m + p) - \hat{w}_{jp}(\omega_p)$, Equation 8 can be written as

$$\lambda_p [w_j(\omega_{m+p}) - w_j(\omega_p)] = i\omega_{m+p} w_j(\omega_{m+p}) - i\omega_p w_j(\omega_p) - i(\omega_{m+p} - \omega_p) \hat{w}_{jp}(\omega_{m+p}) \quad (9)$$

Also, if $\omega_m + p - \omega_p$ is small and $\hat{w}_{jp}(\omega_m + p)$ is not too large, we may neglect the last term of Equation 9 and we have as the equation for an initial value for λ_p

$$\lambda_p = i[\omega_{m+p} w_j(\omega_{m+p}) - \omega_p w_j(\omega_p)] / [w_j(\omega_{m+p}) - w_j(\omega_p)] \quad (10)$$

Next, neglecting the term $\hat{w}_{jp}(\omega)$ in Equation 6 the starting value for b_{jp} is given by

$$b_{jp} = (i\omega_p - \lambda_p) w_j(\omega_p) \quad (11)$$

Thus, from Equations 10 and 11 we can determine initial approximations λ_p and b_{jp} for $p=1, \dots, m$ and for $j=1, \dots, m$. These initial values depend on the function $w_j(\omega)$ at n points ω_p and $\omega_m + p$ for $p=1, \dots, m$. Once these initial values are determined, they can be used in the right hand side of the exact equations, namely

$$\lambda_p = \frac{i[\omega_{m+p} (w_j(\omega_{m+p}) - \hat{w}_{jp}(\omega_{m+p})) - \omega_p (w_j(\omega_p) - \hat{w}_{jp}(\omega_p))]}{[w_j(\omega_{m+p}) - \hat{w}_{jp}(\omega_{m+p}) - w_j(\omega_p) + \hat{w}_{jp}(\omega_p)]} \quad (12)$$

and

$$b_{jp} = (w_j(\omega_p) - \hat{w}_{jp}(\omega_p)) (i\omega_p - \lambda_p) \quad (13)$$

to obtain better approximations for λ_p and b_{jp} for $p=1, \dots, m$ and $j=1, \dots, m$. The values of λ_p and b_{jp} so obtained can then be used again in Equations 12 and 13 to compute still better values of λ_p and b_{jp} and so on until prescribed tolerances are satisfied (or a fixed maximum number of iterations completed).

Performing the computations described, Equations 10, 11, 12 and 13 determine vectors b_k , $k=1, \dots, m$ where b_k has components b_{jk} for $j=1, \dots, m$. Hence from Equation 2 we have

$$u_k v_k^T r = b_k \quad (14)$$

For the case where the matrices M , C and K are symmetric, excitation at a single point is all that is needed to determine the vectors u_k and v_k ; since v_k is just some scalar multiple of u_k . Thus we may take

$$r = \hat{r} e_1 \quad (15)$$

where e_1 denotes the coordinate vector with first component 1 and all other components zero. Moreover, we may suppose the vectors u_k normalized so that the first component of u_k has the value 1.

$$u_{1k} = 1 \quad (16)$$

Then from Equation 14 we have

$$v_k^T r = \hat{r} v_{1k} = b_{1k}$$

or

$$v_{1k} = b_{1k} / \hat{r} \quad (17)$$

and for $j \neq 1$

$$u_{jk} = b_{jk} / b_{1k} \quad (18)$$

and

$$v_{jk} = v_{1k} \cdot u_{jk} \quad (19)$$

In order to describe the process for the nonsymmetric case, it is convenient to modify slightly the notation used in the symmetric case. Let us denote the vectors b_k by $b_k(1)$. The 1 in parenthesis indicates that the vectors $b_k(1)$ are determined from data obtained by exciting Equation 1 at station 1. If we had excited Equation 1 at station q with the same set of frequencies as used at station 1, then from the associated response we could have determined vectors $b_k(q)$ in exactly the same way

(using the same equations) as the vectors $b_k(1)$ were determined. However, we will see that it is not necessary to determine the vectors $b_k(q)$ for $k=1, \dots, m$, and $q=1, \dots, m$ completely.

Nevertheless, for the moment suppose that at the same time λ_k and $b_k(1)$ are being determined iteratively we compute also $b_k(2), \dots, b_k(m)$ for $k=1, \dots, m$. For the nonsymmetric case only Equation 19 does not hold but from Equations 16, 17 and 18 we determine from the vectors $b_k(1)$ the vectors u_k and the component v_{1k} of v_k for $k=1, \dots, m$. Thus we need the remaining components of the vectors v_k .

For the vectors $b_k(q)$, for $k=1, \dots, m$ we have the relation

$$u_k v_k^T e_q \hat{r} = u_k v_{qk} \hat{r} = b_k(q) \quad (20)$$

In this vector equation there is only one unknown, namely, the scalar v_{qk} . From this it is clear that all we need to determine the scalar v_{qk} is one component of the vector $b_k(q)$ for $k=1, \dots, m$.

In order to summarize the information required for the nonsymmetric case let us rewrite Equation 2 as

$$y(\omega, q) = \sum_{k=1}^n \frac{u_k v_k^T e_q \hat{r}}{i\omega - \lambda_k} \quad (21)$$

Let $w_j(\omega, q)$ denote the j th component of the vector $y(\omega, q)$. Then in place of Equation 3 we have

$$w_j(\omega, q) = \sum_{k=1}^m (b_{jk}(q)/(i\omega - \lambda_k) + \bar{b}_{jk}(q)/(i\omega - \bar{\lambda}_k)) \quad (22)$$

For the nonsymmetric case we need the values $w_j(\omega_p, 1)$ and $w_j(\omega_m + p, 1)$ for $p=1, \dots, m$ for each value of j for $j=1, \dots, m$. We need also $w_j(\omega_p, q)$ and $w_j(\omega_m + p, q)$ for $p=1, \dots, m$ for each value of q for $q=2, \dots, m$ but only for one value of j .

As in the symmetric case set

$$\hat{w}_{jp}(\omega, q) = \bar{b}_{jp}(q)/(i\omega - \bar{\lambda}_p) + \sum_{\substack{k=1 \\ k \neq p}}^m (b_{jk}(q)/(i\omega - \lambda_k) + \bar{b}_{jk}(q)/(i\omega - \bar{\lambda}_k)) \quad (23)$$

Then

$$b_{jp}(q)/(i\omega - \lambda_p) = w_j(\omega, q) - \hat{w}_{jp}(\omega, q) \quad (24)$$

And we obtain therefrom

$$(i\omega_p - \lambda_p)[w_j(\omega_p, q) - \hat{w}_{jp}(\omega_p, q)] = (i\omega_{m+p} - \lambda_p)[w_j(\omega_{m+p}, q) - \hat{w}_{jp}(\omega_{m+p}, q)] \quad (25)$$

The starting values for λ_k are computed from

$$\lambda_p = i[\omega_{m+p} w_j(\omega_{m+p}, 1) - \omega_p w_j(\omega_p, 1)] / [w_j(\omega_{m+p}, 1) - w_j(\omega_p, 1)] \quad (26)$$

for any convenient value of j for $p=1, \dots, m$. The starting values for $b_{jp}(1)$ are computed from

$$b_{jp}(1) = (i\omega_p - \lambda_p) w_j(\omega_p, 1) \quad (27)$$

for $p=1, \dots, m$ and for $j=1, \dots, m$. Starting values for $b_{jp}(q)$ are computed from

$$b_{jp}(q) = (i\omega_p - \lambda_p) w_j(\omega_p, q) \quad (28)$$

for say $j=1$ and for $p=1, \dots, m$ and $q=2, \dots, m$.

Once we have approximations for λ_k and $b_{jk}(q)$, the function $\hat{w}_{jp}(\omega, q)$ can be evaluated at ω_p and ω_{m+p} . Improved values of λ_k can then be computed from the equation

$$\lambda_p = \frac{i[\omega_{m+p}(w_j(\omega_{m+p}, 1) - \hat{w}_{jp}(\omega_{m+p}, 1)) - \omega_p(w_j(\omega_p, 1) - \hat{w}_{jp}(\omega_p, 1))]}{[w_j(\omega_{m+p}, 1) - \hat{w}_{jp}(\omega_{m+p}, 1) - w_j(\omega_p, 1) + \hat{w}_{jp}(\omega_p, 1)]} \quad (29)$$

for any convenient value of j and for $p=1, \dots, m$. Improved values of $b_{jp}(1)$ are computed from the equation

$$b_{jp}(1) = (w_j(\omega_p, 1) - \hat{w}_{jp}(\omega_p, 1))(i\omega_p - \lambda_p) \quad (30)$$

for $p=1, \dots, m$ and $j=1, \dots, m$. Similarly, improved values of $b_{jp}(q)$ are obtained from

$$b_{jp}(q) = (w_j(\omega_p, q) - \hat{w}_{jp}(\omega_p, q))(i\omega_p - \lambda_p) \quad (31)$$

for say $j=1, p=1, \dots, m$ and $q=2, \dots, m$. The values of λ_p and $b_{jp}(q)$ obtained from Equations 29, 30, and 31 are used again in these equations to obtain improved values and so on until tolerance requirements are satisfied or a fixed number of iterations completed.

From the values of $b_{jk}(q)$ obtained from the above computations we now compute

$$v_{1k} = b_{1k}(1)/\hat{r} \quad (32)$$

for $k=1, \dots, m$. Next

$$u_{jk} = b_{jk}(1)/b_{1k}(1) \quad (33)$$

for $j=2, \dots, m$ and $k=1, \dots, m$. Recall

$$u_{1k} = 1 \quad (34)$$

for $k=1, \dots, m$. Then

$$v_{qk} = b_{1k}(q)/\hat{r} \quad (35)$$

for $q=2, \dots, m$ and $k=1, \dots, m$. It is clear that the value of \hat{r} in Equations 32 and 35 could be different for different values of the index q .

SECTION III

TWO CLOSE CHARACTERISTIC VALUES

In this section we describe a modification of the procedure given in Section II which enables us to handle the case of two close characteristic values. In order to describe this procedure we consider a complex valued function $w(\omega)$ of the form

$$w(\omega) = \sum_{k=1}^2 (b_k/(i\omega - \lambda_k) + \bar{b}_k/(i\omega - \bar{\lambda}_k)) \quad (36)$$

The problem is, as in Section II, the determination of the λ_k and b_k ; but complicated by the fact that λ_1 and λ_2 are close. Hence, for example we may not neglect the term $b_2/(i\omega - \lambda_2)$ when ω is close to $\text{Im}[\lambda_1]$.

First, we want to make clear the relation of the function $w(\omega)$ of Equation 36 to the frequency response function and the iteration process of Section II. Thus $w(\omega)$ denotes the value of the j th component of the frequency response function, that is, the function $w_j(\omega)$ of Equation 3 with all the terms for $k > 2$ either ignored as a first approximation in the iteration process or subtracted off for the succeeding steps of the iteration process. The subscript j has been left off for notational simplicity.

Set

$$\hat{w}(\omega) = w(\omega) - \bar{b}_1/(i\omega - \bar{\lambda}_1) - \bar{b}_2/(i\omega - \bar{\lambda}_2) \quad (37)$$

that is

$$\hat{w}(\omega) = b_1/(i\omega - \lambda_1) + b_2/(i\omega - \lambda_2) \quad (38)$$

We suppose $w_j = w(\omega_j)$ is known for four values of ω_j close to $\text{Im}[\lambda_1]$ and to $\text{Im}[\lambda_2]$. Using this data, Equation 38 becomes a system of four equations for the four unknowns b_1 , b_2 , λ_1 and λ_2 . The unknowns b_1 and b_2 are easily eliminated from this system. In doing so one obtains two equations for λ_1 and λ_2 . We will now give some of the details.

Solve Equation 38 for b_1

$$b_1 = \hat{w}_j(i\omega_j - \lambda_1) - b_2(i\omega_j - \lambda_1)/(i\omega_j - \lambda_2) \quad (39)$$

then subtract Equation 39 for $j \neq 4$ from Equation 39 with $j = 4$ and after some rearranging and simplifying obtain

$$ib_2(\lambda_1 - \lambda_2) = [i(\omega_j \hat{w}_j - \omega_4 \hat{w}_4) - \lambda_1(\hat{w}_j - \hat{w}_4)]P(2, j, 4)/(\omega_j - \omega_4) \quad (40)$$

where

$$P(k, j, \ell) = \lambda_k^2 - i(\omega_j + \omega_\ell) - \omega_j \omega_\ell \quad (41)$$

for $\ell = 4$ and $j = 1, 2, 3$; that is, a system of three equations for the unknowns b_2 , λ_1 and λ_2 . Clearly, b_2 is readily eliminated from Equation 40 leaving us with a system of two equations which we can write in the form

$$\lambda_1 \left[\frac{(\hat{w}_3 - \hat{w}_4)P(2, 3, 4)}{\omega_4 - \omega_3} - \frac{(\hat{w}_j - \hat{w}_4)P(2, j, 4)}{\omega_4 - \omega_j} \right] = \quad (42)$$

$$i \left[\frac{(\omega_3 \hat{w}_3 - \omega_4 \hat{w}_4)P(2, 3, 4)}{\omega_4 - \omega_3} - \frac{(\omega_j \hat{w}_j - \omega_4 \hat{w}_4)P(2, j, 4)}{\omega_4 - \omega_j} \right]$$

for $j = 1, 2$.

Observe that λ_1 occurs linearly in this system of equations. Hence λ_1 could be eliminated leaving a single equation of the fourth degree in λ_2 . This resulting polynomial equation could be solved for λ_2 and, of course, we would have to choose the correct root. Thus taking $\hat{w}_j = w(\omega_j)$ as a first approximation we could determine λ_2 as described, compute λ_1 from Equation 42, b_2 from Equation 40 and b_1 from Equation 39. In this way one obtains starting values for b_1 , b_2 , λ_1 and λ_2 . These initial

values for b_1 , b_2 , λ_1 and λ_2 can be used in Equation 37 to compute better values of \hat{w}_j for $j = 1, 2, 3, 4$ and the procedure just described for calculating λ_2 , λ_1 , b_1 and b_2 repeated.

Since solving a fourth degree polynomial for λ_2 and selecting the appropriate root is not an attractive procedure, we sought an alternative. Let us make two observations. First, if λ_2 is known then Equation 42 uniquely determines λ_1 . Secondly, if we had first eliminated b_2 and then b_1 we would have obtained Equation 41 with the roles of λ_1 and λ_2 interchanged; so if we know λ_1 then λ_2 is uniquely determined. All the equations needed for treating the case of two close characteristic values are now available.

We do not feel that we have sufficient numerical experience to give a rigidly fixed recipe. However, of the various alternatives which we tried the procedure which we describe next seemed best. We assumed an initial approximation to λ_2 . From Equation 41 we compute

$$P(2, j, 4) = \lambda_2^2 - i(\omega_j + \omega_4)\lambda_2 - \omega_j\omega_4 \quad (43)$$

for $j = 1$ and 3 . Then from Equation 42 we have

$$\lambda_1 \left[\frac{(\hat{w}_3 - \hat{w}_4)P(2, 3, 4)}{\omega_4 - \omega_3} - \frac{(\hat{w}_1 - \hat{w}_4)P(2, 1, 4)}{\omega_4 - \omega_1} \right] =$$

$$i \left[\frac{(\omega_3 \hat{w}_3 - \omega_4 \hat{w}_4)P(2, 3, 4)}{\omega_4 - \omega_3} - \frac{(\omega_1 \hat{w}_1 - \omega_4 \hat{w}_4)P(2, 1, 4)}{\omega_4 - \omega_1} \right] \quad (44)$$

which is easily solved for λ_1 . Now that λ_1 and λ_2 are known we use Equation 40 to compute b_2 , Equation 39 to compute b_1 and Equation 37 to obtain improved values of \hat{w}_j for $j = 1, 2, 3$, and 4 .

Next we compute

$$P(1, j, 1) = \lambda_1^2 - i(\omega_j + \omega_1) \lambda_1 - \omega_j \omega_1 \quad (45)$$

for $j = 2, 4$ and then solve

$$\lambda_2 \left[\frac{(\hat{w}_2 - \hat{w}_1)P(1, 2, 1)}{\omega_1 - \omega_2} - \frac{(\hat{w}_4 - \hat{w}_1)P(1, 4, 1)}{\omega_1 - \omega_4} \right] = \quad (46)$$

$$i \left[\frac{(\omega_2 \hat{w}_2 - \omega_1 \hat{w}_1)P(1, 2, 1)}{\omega_1 - \omega_2} - \frac{(\omega_4 \hat{w}_4 - \omega_1 \hat{w}_1)P(1, 4, 1)}{\omega_1 - \omega_4} \right]$$

for λ_2 . Now that a new value of λ_2 is available we use

$$ib_1(\lambda_2 - \lambda_1) = [i(\omega_j \hat{w}_j - \omega_1 \hat{w}_1) - \lambda_2(\hat{w}_j - \hat{w}_1)] P(1, j, 1) / (\omega_j - \omega_1) \quad (47)$$

with either $j = 2$ or $j = 4$ to compute b_1 and

$$b_2 = \hat{w}_j(i\omega_j - \lambda_2) - b_1(i\omega_j - \lambda_2) / (i\omega_j - \lambda_1) \quad (48)$$

again with either $j = 2$ or $j = 4$.

In the procedure described above we have assumed

$$0 < \text{Im}[\lambda_1] < \text{Im}[\lambda_2]$$

and

$$\omega_1 < \omega_3 < \omega_2 < \omega_4 .$$

It also seemed better to perform two successive updates of the close characteristic values and their associated coefficients for every update of the well separated characteristic values and their associated coefficients, at least in the later stages of the iteration process.

SECTION IV

THE FREQUENCY RESPONSE FUNCTION BY NEWTON'S METHOD

In Section II we described the determination of the frequency response function from experimental data by an iteration process. In this section we describe the determination of the frequency response function by Newton's (iteration) method. The method of Section II and the Newton method are not too different from one another.

For simplicity we leave off the subscript denoting the component of the frequency response function (Equation 3). The problem is, as in Section II, the determination of the unknowns b_k and λ_k in the function.

$$w(\omega) = \sum_{k=1}^m [b_k/(i\omega - \lambda_k) + \bar{b}_k/(i\omega - \bar{\lambda}_k)] \quad (49)$$

from values $w_j = w(\omega_j)$ for $j=1, \dots, n=2m$.

Set

$$W(\omega, b_1, \lambda_1, \dots, b_m, \lambda_m) = \sum_{k=1}^m b_k/(i\omega - \lambda_k) \quad (50)$$

$W(\omega, b_1, \lambda_1, \dots, b_m, \lambda_m)$ is a function of the variables $\omega, b_1, \lambda_1, \dots, b_m, \lambda_m$. Suppose we know

$$W_j = W(\omega_j, b_1, \lambda_1, \dots, b_m, \lambda_m) \quad (51)$$

for n values of j and we have some initial "guess" $b_1^0, \lambda_1^0, \dots, b_m^0, \lambda_m^0$ for the values $b_1, \lambda_1, \dots, b_m, \lambda_m$. Then from Equation 50 we can compute

$$W_j^0 = W(\omega_j, b_1^0, \lambda_1^0, \dots, b_m^0, \lambda_m^0) \quad (52)$$

for $J=1, \dots, 2n$ and from the system of equations

$$\frac{\partial W}{\partial b_1} db_1 + \frac{\partial W}{\partial \lambda_1} d\lambda_1 + \dots + \frac{\partial W}{\partial b_m} db_m + \frac{\partial W}{\partial \lambda_m} d\lambda_m = W_j - W_j^0 \quad (53)$$

for $j=1, \dots, n$ we can determine the increments $db_1, d\lambda_1, \dots, db_m, d\lambda_m$ for improving the approximations $b_1^o, \lambda_1^o, \dots, b_m^o, \lambda_m^o$. Thus

$$b_k^o(\text{new}) = b_k^o + db_k$$

and

$$\lambda_k^o(\text{new}) = \lambda_k^o + d\lambda_k$$

(54)

In the system of equations defined by Equation 53

$$\partial W / \partial b_k = 1 / (i\omega_j - \lambda_k^o) \quad (55)$$

and

$$\partial W / \partial \lambda_k = b_k^o / (i\omega_j - \lambda_k^o)^2 \quad (56)$$

In the practical situation we do not have the function values W_j but have instead the function values w_j . Hence we replace W_j in the system Equation 53 by

$$W_j = w_j - \sum_{k=1}^m \bar{b}_k^o / (i\omega_j - \bar{\lambda}_k^o) \quad (57)$$

The initial guess for the b_k and λ_k is obtained in the same way as in Section II. The main difficulty with this process is solving the n by n system of complex equations, Equation 53, for each iteration step. At this time we believe the method of Section II will involve fewer computations even though it may require more iterations. The Newton process given here is for the case of well separated characteristic values only.

Here now we re-examine briefly the procedure presented in Section II. We can simplify the notation and still illustrate the main features if we consider, as above Equation 49, a single complex valued function. We suppose the value $w(\omega)$ given for $2m$ values of ω . Thus we take

$$w(\omega_j) = w_j \quad (58)$$

$$w(\omega_{m+j}) = w_{m+j}$$

for $j = 1, \dots, m$. We suppose

$$\omega_j < \text{Im} [\lambda_j] < \omega_{m+j} \quad (59)$$

and that the difference $\omega_{m+j} - \omega_j$ is small. The problem is the same as above, namely, solving the resulting system of equations for the m coefficients b_k and the m characteristic values λ_k .

For the $2m$ values of ω the associated system of equations, Equation 49, is nonlinear in the λ_k . The first step in the solution process is to rewrite this system of equations. Set

$$W_j = \bar{b}_j / (i\omega_j - \bar{\lambda}_j) + \sum_{\substack{k=1 \\ k \neq j}}^m [b_k / (i\omega_j - \lambda_k) + b_k / (i\omega_j - \bar{\lambda}_k)] \quad (60)$$

W_{m+j} is defined similarly.

The system of Equation 49 now appears as

$$\begin{aligned} w_j &= b_j / (i\omega_j - \lambda_j) + W_j \\ w_{m+j} &= b_j / (i\omega_{m+j} - \lambda_j) + W_{m+j} \end{aligned} \quad (61)$$

for $j=1, \dots, m$. For a fixed value of j the two equations in the system, Equation 61, can be solved for λ_j and b_j . We have

$$\begin{aligned} \lambda_j &= i[\omega_{m+j}W_{m+j} - \omega_jW_j - (\omega_{m+j}W_{m+j} - \omega_jW_j)] / [w_{m+j} - w_j - (W_{m+j} - W_j)] \\ b_j &= (w_j - W_j)(i\omega_j - \lambda_j) \end{aligned} \quad (62)$$

for $j = 1, \dots, m$. Now then let us regard the λ_j and b_j as variables. If we substitute in the right hand side of Equation 62 values λ_{jo}, b_{jo} for $j = 1, \dots, m$ we will obtain say values λ_{j1}, b_{j1} , for $j = 1, \dots, m$. Thus the system of Equation 62 transforms a set of $2m$ complex numbers into a set of $2m$ complex numbers. The desired solution is a set of $2m$ complex numbers which transform into the same set of complex numbers respectively.

In Section II we take $b_1 = \dots = b_m = 0$ initially. For this particular assignment of the coefficients b_k it is not necessary to specify the λ_k . Then for $j=1$ we use Equation 62 to compute λ_1 and b_1 . We could have computed λ_j and b_j for $j = 2, \dots, m$ also for $b_1 = \dots = b_m = 0$. Instead, we use λ_1, b_1 and $b_2 = \dots = b_m = 0$ to compute λ_2 and b_2 and so on.

We would like to be able to show that the transformation defined by Equation 62 is a contraction mapping. Then it would follow that the iteration process of Section II converges to a unique fixed point. As of this time, however, we have not been able to determine a region of convergence.

SECTION V

RESULTS AND CONCLUSIONS

In this section we describe and discuss briefly the numerical experiments performed to test the procedures presented in Sections II and III. From the discussion there and Appendix A the problem of interest reduces to the determination of parameters b_k and λ_k which define a complex valued function $w(\omega)$ of the form

$$w(\omega) = \sum_{k=1}^m [b_k/(i\omega - \lambda_k) + \bar{b}_k/(i\omega - \bar{\lambda}_k)]$$

Our first objective is to exhibit the influence of a coefficient b_k , in our case b_4 , on the function $w(\omega)$. The characteristic values λ_k and coefficients b_k used for defining functions $w(\omega)$ are listed in Tables 1A-4A. The values of λ_k and b_k are the same in all these tables except for b_4 . Since λ_k and b_k are complex quantities it takes a pair of real numbers to specify them. The functions $w(\omega)$ resulting from simple modifications of the coefficient b_4 are shown graphically in Figures 7a-10a. Figures 7b-10b are the graphs of $|w(\omega)|$ for the same modifications of the coefficient b_4 . In order to facilitate comparison these graphs are shown collectively in Figures 6a and 6b respectively.

One observes, Figure 6a, as the coefficient b_4 changes from quadrant to quadrant there is a corresponding change in one loop of the associated graph. The remaining portion of the graph is not visibly changed. Thus it is clear that the term $b_k/(i\omega - \lambda_k)$ is the principal contributor to the value of the function $w(\omega)$ when ω is in the vicinity of $I_m[\lambda_k]$. This feature is illustrated further in Figures 7a-10a where the x's are the values of the term $b_4/(i\omega - \lambda_4)$ alone as ω varies from 4.3 to 5.7 at steps of 0.05.

In Table 1A, for example, twelve frequencies (in two columns) are listed. The corresponding function values of $w(\omega)$ are listed immediately thereafter. These function values are indicated on the corresponding graphs, Figures 7a and 7b, by the "boxes". From these twelve function values, using the method of Section II, the characteristic values λ_k and

coefficients b_k are determined. The characteristic values and coefficient computed from the given function values are listed in Table 1B. The "radius of differences" is the square root of the sum of squares of the real and imaginary parts of the differences of the last two successive calculations of the characteristic values.

Table 1A also gives "truncated" values of the function $w(\omega)$ at the listed frequencies. Table 1C gives the characteristic values and coefficients obtained from computations using these truncated values. Thus Table 1C shows how much the lack of precision in the experimental data affects the computations. Tables 2A-4C show the results as the coefficient b_4 is changed. As one would expect the method of Section II is not sensitive to changes in the coefficient b_k .

Our second objective is to test the procedures described in Section III for two close characteristic values. The data and results for this set of problems is given in the Tables 5A-9D. The coefficient b_k and characteristic values λ_k , except for λ_6 , are the same throughout the set of problems. The characteristic value λ_6 gets close to λ_5 by letting $I_m[\lambda_6]$ take on the values 4.5, 4.3, 4.2, 4.1 and 4.05 successively.

In order to see the effect of λ_6 tending to λ_5 the graphs, Figures 12a-15a and 12b-15b are collected in Figures 11a and 11b respectively. One observes, in Figure 11a, that as λ_6 tends to λ_5 the "circle" corresponding to λ_5 deteriorates, diminishes and finally disappears. At the same time the "circle" corresponding to λ_6 distorts and expands and eventually becomes nearly circular again when λ_6 is practically the same as λ_5 . Similarly, the valley between the peaks associated with λ_5 and λ_6 (Figure 11b) disappears as λ_6 tends to λ_5 and the two peaks become a single high peak when λ_6 is practically the same as λ_5 .

It is clear that the behavior just described depends upon the coefficients b_5 and b_6 . To illustrate the role of b_5 and b_6 envision what takes place as λ_6 tends to λ_5 for the case $b_6 = -b_5$.

In Tables 5B-9B we give the results using the procedures described in Sections II and III and the exact function values. In Tables 5C-9C we give the results when the truncated function values were used. We observe that with accurate data all coefficients b_k are computed well. With inaccurate data, the difference between the actual and computed coefficients gets large particularly for the differences corresponding to λ_5 and λ_6 respectively as λ_6 tends to λ_5 .

Using the computed characteristic values and coefficients from Table 9C we computed the function values at the given set of frequencies. These computed function values are listed in Table 9D. Observe that the computed function values in Table 9D and the truncated function values in Table 9C agree very well. Observe also that graphically, in Figure 17, the function values based on the computed coefficients from Table 9C are indistinguishable from function values based on the exact characteristic values and coefficient.

We believe the numerical experiments performed in Reference 1 and this report show that the procedure, described in Section II of Reference 1 and Sections II and III of this report, is a highly accurate method for determining the complex characteristic values and associated complex characteristic vectors of a light, viscously damped linear system from the responses to sinusoidal excitations. The method as presented in Reference 1 was capable of handling well separated characteristic values and characteristic values of multiplicity great than 1.

In Reference 1 we tentatively assumed that the characteristic values could be regarded either as well separated or identical. However, we were concerned that this assumption might not be realistic. Accordingly we extended the method, Section III, so that it could handle the case of two characteristic values which were too close for the well separated procedure yet not close enough to be regarded as identical.

Further numerical experiments should be performed to determine more precisely when two characteristic values should no longer be regarded as well separated but rather as close and when no longer as close but identical. We need also to perform numerical experiments to determine how many characteristic values the method can handle. It is also of interest to determine the degree of damping at which the method is no longer reliable.

APPENDIX A

LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Methods for determining vibration parameters are based on facts and relations from the theory of linear systems of ordinary differential equations with constant coefficients. These facts and relations are well known, at least to the expert. In this appendix, it is our objective to develop in a brief, yet complete fashion, these relations so that they are readily accessible and acceptable.

Consider a second order system of differential equations

$$M\ddot{x} + C\dot{x} + Kx = f \quad (A1)$$

Here M , C and K denote real matrices of order m and $x = x(t)$ and $f = f(t)$ are m -dimensional vector functions of t . The vector functions $x(t)$ and $f(t)$ may have complex components.

The associated homogeneous equation

$$M\ddot{x} + C\dot{x} + Kx = 0 \quad (A2)$$

has nontrivial solutions of the form $u \exp(\lambda t)$ if and only if λ satisfied the characteristic equation

$$\det[M\lambda^2 + C\lambda + K] = 0 \quad (A3)$$

and the vector u satisfies the condition

$$[M\lambda^2 + C\lambda + K] u = 0 \quad (A4)$$

Similarly, if a vector v satisfies the condition

$$[M^T\lambda^2 + C^T\lambda + K^T] v = 0 \quad (A5)$$

then $v \exp(\lambda t)$ satisfies the transposed homogeneous equation

$$M^T\ddot{x} + C^T\dot{x} + K^Tx = 0 \quad (A6)$$

Throughout this appendix the symbol v is reserved for quantities associated with the transposed equation. We will refer to values of λ which satisfy the characteristic equation, Equation A3, as characteristic values and the vectors u and v as corresponding (or associated) characteristic vectors.

The characteristic equation is a polynomial equation in λ . Since the coefficients are real, the complex conjugate of any complex root is also a root. We suppose Equation A3 is of degree $n = 2m$ and that all the roots are complex with negative real part. We suppose also that if λ is a root of multiplicity p then the matrix $[M\lambda^2 + C\lambda + K]$ is of rank $m-p$.

For convenience we index the roots of Equation A3 so that $0 < \text{Im}[\lambda_1] \leq \text{Im}[\lambda_2] \leq \dots \leq \text{Im}[\lambda_m]$ and $\bar{\lambda}_{m+k} = \lambda_k$, for $k = 1, \dots, m$. Let u_k and v_k denote characteristic vectors corresponding to λ_k for any value of $k \leq m$. That is, u_k and v_k satisfy the conditions expressed by Equations A4 and A5 respectively. Then $u_{m+k} = \bar{u}_k$ and $v_{m+k} = \bar{v}_k$ are characteristic vectors corresponding to $\lambda_{m+k} = \bar{\lambda}_k$.

Orthogonality conditions are well known and convenient normalizing relations are readily determined for first order systems of differential equations. For these reasons we consider the first order system

$$\dot{x}_1 - x_2 = 0 \quad (A7)$$

$$M\dot{x}_2 + Kx_1 + Cx_2 = f$$

which is a system equivalent to Equation A1. This system may be written in block matrix form as

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad (A8)$$

and symbolically as

$$A\dot{y} + By = g \quad (A9)$$

One can readily verify $\tilde{u}_k \exp(\lambda_k t)$, where

$$\tilde{u}_k = \begin{bmatrix} u_k \\ \lambda_k u_k \end{bmatrix} \quad (A10)$$

satisfies the homogeneous equation $A\dot{y} + By = 0$. Similarly, $\tilde{v}_k \exp(\lambda_k t)$

$$\tilde{v}_k = \begin{bmatrix} (-1/\lambda_k) K^T v_k \\ v_k \end{bmatrix} \quad (A11)$$

satisfies the transposed homogeneous equation $A^T \dot{y} + B^T y = 0$. The vectors u_k and v_k are characteristic vectors satisfying Equations A4 and A5 respectively.

From the equations $\lambda_k A \tilde{u}_k + B \tilde{u}_k = 0$ and $\lambda_j \tilde{v}_j^T A + \tilde{v}_j^T B = 0$ it follows that

$$\lambda_k \tilde{v}_j^T A \tilde{u}_k + \tilde{v}_j^T B \tilde{u}_k = 0$$

$$\lambda_j \tilde{v}_j^T A \tilde{u}_k + \tilde{v}_j^T B \tilde{u}_k = 0$$

On subtracting the second of these two equations from the first one finds for $j \neq k$, if $\lambda_j \neq \lambda_k$ that the characteristic vectors \tilde{u}_k and \tilde{v}_j satisfy the orthogonality conditions

$$\tilde{v}_j^T A \tilde{u}_k = 0 \text{ and } \tilde{v}_j^T B \tilde{u}_k = 0 \quad (A12)$$

For a characteristic value λ_k of multiplicity $p > 1$, the characteristic vectors can be determined in pairs $\tilde{u}_k, \tilde{v}_k, \tilde{u}_{k+1}, \tilde{v}_{k+1}, \dots, \tilde{u}_{k+p}, \tilde{v}_{k+p}$ so that the orthogonality conditions expressed by Equations A12 hold whenever $j \neq k$.

For reasons which will be apparent shortly, we suppose the vectors \tilde{u}_k and \tilde{v}_k normalized so that

$$\tilde{v}_k^T A \tilde{u}_k = 1 \quad (A13)$$

then, it follows that

$$\tilde{v}_k^T B \tilde{u}_k = -\lambda_k \quad (A14)$$

Let U denote the matrix whose columns are the characteristic vectors \tilde{u}_k . Thus

$$U = [\tilde{u}_1 \dots \tilde{u}_n] \quad (A15)$$

and similarly

$$V = [\tilde{v}_1 \dots \tilde{v}_n] \quad (A16)$$

Also let Λ denote the diagonal matrix having the characteristic values $\lambda_1, \dots, \lambda_n$ on the main diagonal. From the orthogonality and normalizing conditions we have

$$V^T A U = I \text{ and } V^T B U = -\Lambda \quad (A17)$$

Let $\Lambda_e = \Lambda_e(t)$ denote the diagonal matrix having $\exp(\lambda_1 t), \dots, \exp(\lambda_n t)$ for its diagonal elements. Set

$$Y(t) = U \Lambda_e(t)$$

Then one recognizes that $A\dot{Y} + BY = 0$ and also that $[AU\Lambda + BU] = 0$. These facts are useful in determining a particular solution of the inhomogeneous equation $A\dot{y} + By = g$. Thus we set $y(t) = Y(t)z(t)$ and substitute this expression for $y(t)$ into Equation A9 we find that

$$AY\dot{z} = AU\Lambda_e\dot{z} = g$$

Using $V^T A U = I$ from Equation A17 we obtain as an expression for $y(t)$

$$y(t) = U \Lambda_e(t) \int_0^t \Lambda^{-1}(\tau) V^T g(\tau) d\tau \quad (A18)$$

It is clear that $y(0)$ and easily verified that $y(t)$ satisfies Equation A9.

In Equation A18, use Equations A15 and A16 to replace U and V^T respectively. Then use Equation A10 to replace the column vectors \tilde{u}_k and

Equation A11 to replace the row vectors v_k^T . Lastly, use the right hand side of Equation A8 to replace g . Thus the integrand of Equation A18 becomes

$$\begin{bmatrix} \exp \lambda_1 \tau & & 0 \\ & \ddots & \\ 0 & & \exp \lambda_n \tau \end{bmatrix} \begin{bmatrix} (-1/\lambda_1) v_1^T K & v_1^T \\ & \\ (-1/\lambda_n) v_n^T K & v_n^T \end{bmatrix} \begin{bmatrix} 0 \\ f(\tau) \end{bmatrix}$$

Performing the indicated matrix-vector multiplications this integrand is rewritten as the vector

$$\begin{bmatrix} (\exp \lambda_1 \tau) v_1^T f(\tau) \\ \vdots \\ (\exp \lambda_n \tau) v_n^T f(\tau) \end{bmatrix}$$

Multiplying this vector by the matrix $U \Lambda_e(t)$ Equation A18 can be rewritten as the two vector equations

$$x_1(t) = \sum_{k=1}^n u_k \exp(\lambda_k t) \int_0^t \exp(-\lambda_k \tau) v_k^T f(\tau) d\tau \quad (A19)$$

$$x_2(t) = \sum_{k=1}^n \lambda_k u_k \exp(\lambda_k t) \int_0^t \exp(-\lambda_k \tau) v_k^T f(\tau) d\tau \quad (A20)$$

It follows that $x_1(t)$ and $x_2(t)$ must satisfy Equation A7. From Equation A7 $\dot{x}_1(t) = x_2(t)$. If we compute $\dot{x}_1(t)$ from Equation A19, we find that

$$\dot{x}_1(t) = x_2(t) + \sum_{k=1}^n u_k v_k^T f(t)$$

and it follows, since $f(t)$ is arbitrary, that

$$\sum_{k=1}^n u_k v_k^T = 0 \quad (A21)$$

In a similar way, we find, since $x_1(t)$ must satisfy the condition $M\ddot{x} + C\dot{x} + Kx = f$, that

$$M \cdot \sum_{k=1}^n \lambda_k u_k v_k^T = I \quad (A22)$$

Using Equation A11 we can write Equation A16 as

$$V = \begin{bmatrix} -K^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_1/\lambda_1 & \dots & v_n/\lambda_n \\ v_1 & \dots & v_n \end{bmatrix} \quad (A23)$$

then

$$V^T A U = \begin{bmatrix} v_1^T/\lambda_1 & \dots & v_n^T/\lambda_n \\ v_1^T & \dots & v_n^T \end{bmatrix} \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} u_1 & \dots & u_n \\ \lambda_1 u_1 & \dots & \lambda_n u_n \end{bmatrix} = I \quad (A24)$$

In the same way we have

$$V^T B U = \begin{bmatrix} v_1^T/\lambda_1 & \dots & v_n^T/\lambda_n \\ v_1^T & \dots & v_n^T \end{bmatrix} \begin{bmatrix} 0 & K \\ K & C \end{bmatrix} \begin{bmatrix} u_1 & \dots & u_n \\ \lambda_1 u_1 & \dots & \lambda_n u_n \end{bmatrix} = -\Lambda \quad (A25)$$

Equations A22, A24 and A25 show that the characteristic values $\lambda_1, \dots, \lambda_m$ and appropriately normalized characteristic vectors u_1, \dots, u_m and v_1, \dots, v_m determine the matrices M, C and K . Actually, Equations A22 and A25 determine M, C and K . The Equation A25 is a matrix equation of order $n = 2m$. Other relations are available for determining the matrices M, C and K .

It follows from $V^T A U = I$ that $A U V^T = I$. Using this last equality one can obtain the results expressed by Equations A21 and A22. With Equation A21 and Equation A22 established in this second independent manner, it is a simple task to verify that the time derivative of Equation A19 is Equation A20. It is also readily verified that $x_1(t)$ as given by Equation A19 satisfies Equation A1.

From Laplace transform theory we have for zero initial conditions

$$L\{x(t)\} = [Ms^2 + Cs + K]^{-1} L\{f(t)\} \quad (A26)$$

We also have

$$L\left\{\int_0^t F_1(t-\tau)F_2(\tau)d\tau\right\} = f_1(s) \cdot f_2(s) \quad (A27)$$

where $f_1(s) = L\{F_1(t)\}$ and $f_2(s) = L\{F_2(t)\}$. Applying the formula expressed by Equation A27 to Equation A19 we obtain

$$L\{x_1(t)\} = \sum_{k=1}^n \frac{u_k v_k^T}{s - \lambda_k} L\{f(t)\} \quad (A28)$$

From Equations A26 and A28 we see that the transfer function

$$[Ms^2 + Cs + K]^{-1} = \sum_{k=1}^n \frac{u_k v_k^T}{s - \lambda_k} = H(s) \quad (A29)$$

say.

Next we want to determine some particular solutions of Equation A1 for some simple vector functions $f(t)$. Let e_j denote the vector whose j th component is 1 and all other components zero. Also let $h(t)$ denote the scalar function satisfying the conditions.

$$\begin{aligned} h(t) &= 0 \quad \text{for } t < 0 \\ &= 1 \quad \text{for } t \geq 0 \end{aligned}$$

Set $f(t) = e_j h(t)$ and denote the response to $e_j h(t)$ by $H_j(t)$. Replacing $f(\tau)$ in Equation A19 by $e_j h(\tau)$ one immediately obtains for the step response $H_j(t)$

$$H_j(t) = \sum_{k=1}^n \frac{j_k v_k^T e_j}{\lambda_k} \exp(\lambda_k t) - \sum_{k=1}^n \frac{u_k v_k^T e_j}{\lambda_k} \quad (A30)$$

The impulse response $I_j(t)$ is the derivative of the step response, that is,

$$I_j(t) = \frac{d}{dt} (H_j(t)) = \sum_{k=1}^n u_k v_k^T e_j \exp(\lambda_k t) \quad (A31)$$

Let $f_j(t)$ denote the j th component of the vector forcing function $f(t)$. Then $f(t)$ can be written as

$$f(t) = e_1 f_1(t) + \dots + e_m f_m(t).$$

Replacing $f(\tau)$ in Equation A19 with this expression, we obtain

$$x_1(t) = \sum_{j=1}^m \int_0^t \sum_{k=1}^n \exp \lambda_k (t-\tau) u_k v_k^T e_j f_j(\tau) d\tau$$

From Equation A31 this formula for the response can be written as

$$x_1(t) = \sum_{j=1}^m \int_0^t I_j(t-\tau) f_j(\tau) d\tau \quad (A32)$$

Next we take $f(t) = r \exp(i\omega t)$ where r denotes a real constant vector and obtain from Equation A19

$$x_1(t) = \sum_{k=1}^n \frac{u_k v_k^T r}{i\omega - \lambda_k} \exp(i\omega t) - \sum_{k=1}^n \frac{u_k v_k^T r}{i\omega - \lambda_k} \exp(\lambda_k t) \quad (A33)$$

For complex λ_k with negative real part, the second summation in Eq. A33 goes to zero as t becomes large. Hence

$$y \exp(i\omega t) = \sum_{k=1}^n \frac{u_k v_k^T r}{i\omega - \lambda_k} \exp(i\omega t) \quad (A34)$$

is the steady state response to the harmonic excitation $r \exp(i\omega t)$.

The steady state response to $r \exp(i\omega t)$ is readily obtained from the steady state response to the sinusoidal excitation $r \sin \omega t$. One knows, or can readily show, that

$$[MD^2 + CD + K] \{ \text{Im}[y \exp(i\omega t)] \} = \text{Im}[r \exp(i\omega t)] = r \sin \omega t \quad (\text{A35})$$

Now

$$\text{Im}[y \exp(i\omega t)] = \text{Re}[y] \sin \omega t + \text{Im}[y] \cos \omega t \quad (\text{A36})$$

and if $x(t)$ denotes the steady state response to the $r \sin \omega t$ the system of equations

$$\text{Re}[y] \sin \omega t_1 + \text{Im}[y] \cos \omega t_1 = x(t_1) \quad (\text{A37})$$

$$\text{Re}[y] \sin \omega t_2 + \text{Im}[y] \cos \omega t_2 = x(t_2)$$

is readily solved for $\text{Re}[y]$ and $\text{Im}[y]$ from the recorded values of $x(t)$ at t_1 and t_2 , provided $\sin \omega t_1 \cos \omega t_2 - \cos \omega t_1 \sin \omega t_2 \neq 0$.

From Equation A34

$$y = \sum_{k=1}^n \frac{u_k v_k^T r}{i\omega - \lambda_k} = \left[\sum_{k=1}^n \frac{u_k v_k^T}{i\omega - \lambda_k} \right] r \quad (\text{A38})$$

The matrix on the right hand side of this equation is called the frequency response function. Observe that it differs from the transfer function, Equation A29, only in the term $i\omega$ in the denominator.

From Equation A38 we have

$$\bar{y} = \sum_{k=1}^n \frac{\bar{u}_k \bar{v}_k^T r}{-i\omega - \bar{\lambda}_k} + \sum_{k=1}^m \frac{u_k v_k^T r}{-i\omega - \lambda_k}$$

and from this equation it is clear that

$$\bar{y} = \sum_{k=1}^n \frac{u_k v_k^T r}{-i\omega - \bar{\lambda}_k} \quad (\text{A39})$$

Consider next the undamped homogeneous system of equations

$$\ddot{M}x + Kx = 0 \quad (A40)$$

associated with Equation A1. We now suppose that M and K are symmetric and that the positive definiteness condition satisfied. Then there are values $0 < \omega_1 \leq \dots \leq \omega_m$ and corresponding real vectors $\hat{u}_1, \dots, \hat{u}_m$ satisfying the condition

$$[-\omega_k^2 M + K] \hat{u}_k = 0 \quad (A41)$$

It is readily shown that the vectors \hat{u}_k satisfy the orthogonality conditions

$$\hat{u}_j^T M \hat{u}_k = 0 \text{ and } \hat{u}_j^T K \hat{u}_k = 0 \quad (A42)$$

We will not impose any particular normalization on the vector \hat{u}_k at this time.

Set

$$-i\hat{r}_k \exp(i\omega_k t) = [MD^2 + CD + K] \{\hat{u}_k \exp(i\omega_k t)\}$$

and

$$i\hat{r}_k \exp(-i\omega_k t) = [MD^2 + CD + K] \{\hat{u}_k \exp(-i\omega_k t)\}$$

On adding these two equations one obtains

$$(MD^2 + CD + K) \{\hat{u}_k \cos \omega_k t\} = \hat{r}_k \sin \omega_k t \quad (A43)$$

This equation simplifies to

$$-\omega_k C \hat{u}_k = \hat{r}_k$$

for $k = 1, \dots, m$. This system in matrix form is

$$C[\hat{u}_1 \dots \hat{u}_m] \begin{bmatrix} -\omega_1 & & 0 \\ & \ddots & \\ 0 & & -\omega_m \end{bmatrix} = [\hat{r}_1 \dots \hat{r}_m] \quad (A44)$$

Equation A44 can be solved for C and we have

$$C = [\hat{r}_1 \dots \hat{r}_m] \begin{bmatrix} -1/\omega_1 & & 0 \\ & \ddots & \\ 0 & & -1/\omega_m \end{bmatrix} [\hat{u}_1 \dots \hat{u}_m]^{-1} \quad (\text{A45})$$

Set

$$\hat{U} = [\hat{u}_1 \dots \hat{u}_m] \quad (\text{A46})$$

Because of the orthogonality conditions

$$\hat{U}^T M \hat{U} = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{bmatrix} \quad (\text{A47})$$

A diagonal matrix with diagonal entries a_1, \dots, a_m which must be determined. Similarly

$$\hat{U}^T K \hat{U} = \begin{bmatrix} \omega_1^2 a_1 & & 0 \\ & \ddots & \\ 0 & & \omega_m^2 a_m \end{bmatrix} \quad (\text{A48})$$

Next for some value of $\omega \neq \omega_k$ we have

$$[M\omega^2 + C\omega + K] y \exp(i\omega t) = r \exp(i\omega t)$$

From this equation one obtains, since we may suppose C is known

$$[-\omega^2 M + K]y = r - i\omega Cy$$

Then

$$\hat{U}^T [-\omega^2 M + K] \hat{U} \hat{U}^{-1} y = \hat{U}^T (r - i\omega Cy) = b \text{ say} \quad (\text{A49})$$

Set

$$U^{-1}y = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

then using Equations A47 and A48 we can rewrite Equation A49 as the system of equations

$$(\omega_k^2 - \omega^2) a_k z_k = b_k$$

for $k=1, \dots, m$. Or, on solving for a_k we have

$$a_k = b_k / z_k (\omega_k^2 - \omega^2) \quad (A50)$$

for $k=1, \dots, m$.

If $\exp(i\omega t)$ is the steady state response to the harmonic excitation $r \exp(i\omega t)$ then y and r satisfy the condition

$$[-\omega^2 M + K + i\omega C]y = r \quad (A51)$$

This system of equations can be rewritten as a system of real equations

$$[-\omega^2 M + K] \operatorname{Re}[y] - \omega C \operatorname{Im}[y] = r \quad (A52)$$

$$C\omega \operatorname{Re}[y] + [-\omega^2 M + K] \operatorname{Im}[y] = 0$$

We have also from Equations A41 and A32

$$y = \sum_{k=1}^n \frac{u_k v_k^T r}{i\omega - \lambda_k} = H(i\omega)r \quad (A53)$$

From this equation we have

$$\begin{aligned} \operatorname{Re}[y] &= \operatorname{Re}[H(i\omega)]r \\ \operatorname{Im}[y] &= \operatorname{Im}[H(i\omega)]r \end{aligned} \quad (A54)$$

If we examine Equation A52 and A54 together we are able to observe an important relation between the transfer function (or better the frequency response function) of Equation 1 and the natural modes and natural frequencies of the associated undamped system. Suppose $\omega = \omega_k$ is a characteristic value of $\operatorname{Re}[H(i\omega)]$ and r is a corresponding characteristic vector, that is

$$\det[\operatorname{Re}(H i \omega_k)] = 0$$

and

$$\operatorname{Re}[y] = \operatorname{Re}[H(i\omega_k)]r = 0$$

Now the vector y determined from Equation A54 by r and ω_k must satisfy Equation A52 and, in particular, since $y = i \operatorname{Im}[y]$ it follows that

$$[-\omega_k^2 M + K] y/i = 0$$

Hence y/i and y are characteristic vectors associated with the characteristic value ω_k .

Conversely, if $\omega = \omega_k$ and a real vector which we denote by $\operatorname{Im}[y]$ satisfy the condition

$$[-\omega_k^2 M + K] \operatorname{Im}[y] = 0,$$

then from Equation A52 we infer $\operatorname{Re}[y] = 0$ and $r = -\omega_k C \operatorname{Im}[y]$. Since these quantities must satisfy Equation A54, it follows in particular that r satisfies the homogeneous equation $\operatorname{Re}[H(i\omega_k)] r = 0$. Hence ω_k satisfies $\det[\operatorname{Re}[H(i\omega_k)]] = 0$.

APPENDIX B

INTERPOLATION WITH EXPONENTIAL FUNCTIONS

In this section we give the mathematical background and details of some procedures for determining the "amplitudes" a_k and the "complex frequencies" λ_k of a function $x(t)$ of the form

$$x(t) = \sum_{k=1}^n a_k \exp(\lambda_k t) \quad (B1)$$

from the values of the function. The function $x(t)$ may be either a scalar or vector function of t and, accordingly, the a_k denote either scalar or vector constants. We consider first the case where $x(t)$ and the a_k are scalars.

Set

$$\eta_k = \exp(\lambda_k h) \quad (B2)$$

then for $j = 0, \dots, n$, we have the system of $(n + 1)$ equations

$$x(t + jh) = \sum_{k=1}^n \eta_k^j a_k \exp(\lambda_k t) \quad (B3)$$

which can be written in matrix form as

$$\begin{bmatrix} x(t) & -1 & \dots & -1 \\ x(t+h) & -\eta_1 & \dots & -\eta_n \\ & & \vdots & \\ x(t+nh) & -\eta_1^n & \dots & -\eta_n^n \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \exp(\lambda_1 t) \\ \vdots \\ a_n \exp(\lambda_n t) \end{bmatrix} = 0 \quad (B4)$$

Now the relation expressed by Equation B4 holds if and only if the coefficient matrix is singular, that is, the determinant of the coefficient matrix is zero.

Let A_{j+1} denote the cofactor of the element $x(t+jh)$. Expanding the determinant of the coefficient matrix in terms of the elements of the first column we obtain.

$$x(t+nh) + (A_n/A_{n+1}) x(t+(n-1)h) + \dots + (A_1/A_{n+1}) x(t) = 0 \quad (B5)$$

Hence, a function $x(t)$ as given by Equation B1 satisfies an n th order recursive relation or difference equation of the form given by Equation B5. If in the matrix of Equation B4, we replace $x(t)$, ..., $x(t+nh)$ by $1, \eta, \dots, \eta^n$ respectively, we observe that the coefficients A_{n+1-k}/A_{n+1} are the elementary symmetric (root) functions $p_k(\eta_1, \dots, \eta_n)$.

$$\begin{aligned} p_1 &= -(\eta_1 + \dots + \eta_n) \\ p_2 &= \eta_1\eta_2 + \dots + \eta_1\eta_n + \eta_2\eta_3 + \dots + \eta_2\eta_n + \dots + \eta_{n-1}\eta_n \\ &\vdots \\ p_n &= (-1)^n \eta_1\eta_2 \dots \eta_n \end{aligned} \quad (B6)$$

Set

$$c_j = A_j/A_{n+1} \quad (B7)$$

for $j=1, \dots, n$ and set

$$x_{jk} = x(t_j + (k-1)h) \quad (B8)$$

for $j=1, \dots, n$ and $k=1, \dots, n+1$. Thus, if we know the value of $x(t)$ at times $t_j + (k-1)h$, then from Equation B5 we obtain the system of equations

$$\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} -x_{1, n+1} \\ \vdots \\ -x_{n, n+1} \end{bmatrix} \quad (B9)$$

for the coefficients c_1, \dots, c_n .

Suppose Equation B9 has been solved for the coefficients c_1, \dots, c_n . We can determine next the complex frequencies λ_k . First, one solves the polynomial equation

$$\eta^n + c_n \eta^{n-1} + \dots + c_1 = 0 \quad (B10)$$

for the roots η_1, \dots, η_n . Set $\rho_k = |\eta_k|$ and $\theta_k = \arg(\eta_k)$ where $0 \leq \theta_k < 2\pi$ then

$$\eta_k = \exp[\log \rho_k + i(\theta_k \pm 2j\pi)] \quad (B11)$$

and λ_k is given by the equation

$$h\lambda_k = \log \rho_k + i(\theta_k \pm 2j\pi) \quad (B12)$$

It is clear from Equation B12 that λ_k is not uniquely determined from η_k . However, in the practical situation the appropriate value of λ_k is usually clear.

The only task remaining is the determination of the amplitudes a_k . From the first n equations of the system given by Equation B3 we have

$$\begin{bmatrix} 1 & \dots & 1 \\ \eta_1 & \dots & \eta_n \\ \vdots & & \vdots \\ \eta_1^{n-1} & \dots & \eta_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \exp(\lambda_1 t) \\ a_2 \exp(\lambda_2 t) \\ \vdots \\ a_n \exp(\lambda_n t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t+h) \\ \vdots \\ x(t+(n-1)h) \end{bmatrix} \quad (B13)$$

It is clear that this linear system of equations is solvable for the amplitudes a_k , provided the coefficient matrix is nonsingular. However, the particular form of the coefficient matrix lets one use a special method for solving this linear system of equations. Let A_{jk} denote the cofactor of the (j,k) th element of the Vandermonde determinant. By Cramer's Rule

$$a_j \exp(\lambda_j t) = \left(\sum_{k=1}^n x(t+(k-1)h) A_{kj} \right) / \left(\sum_{k=1}^n \eta_j^{k-1} A_{kj} \right) \quad (B14)$$

Dividing both numerator and denominator of the right hand side of this equation by A_{nj} one observes that the ratios A_{kj}/A_{nj} are the elementary symmetric (root) functions of the η_k 's, excluding η_j . This observation is the basis of the algorithm given in Reference 14 (page 275) for computing the amplitudes a_j . In order to see clearly how this algorithm was obtained we need to make several more observations.

Set

$$\begin{aligned} f(n) &= (n-n_1) \dots (n-n_n) \\ &= n^n + p_1 n^{n-1} + \dots + p_n \end{aligned} \quad (B15)$$

then the denominator

$$\sum_{k=1}^n n_j^{k-1} A_{kj} / A_{kn} = \frac{df}{dn} \big|_{n=n_j} \quad (B16)$$

$$\begin{aligned} f/(n-n_j) &= [f(n)-f(n_j)]/(n-n_j) = \\ &= (n^{n-1} + n^{n-2}n_j + \dots + nn_j^{n-2} + n_j^{n-1}) + p_1(n^{n-2} + n^{n-3}n_j + \dots + n_j^{n-2}) \\ &+ p_2(n^{n-3} + n^{n-4}n_j + \dots + n_j^{n-3}) + \dots + p_{n-1} \end{aligned} \quad (B17)$$

Collecting terms in like powers of n we obtain from Equation B17

$$\begin{aligned} f/(n-n_j) &= n^{n-1} + (n_j + p_1)n^{n-2} + (n_j^2 + p_1n_j + p_2)n^{n-3} + (n_j^3 + p_1n_j^2 + p_2n_j + p_3)n^{n-4} + \\ &\dots + (n_j^{n-1} + p_1n_j^{n-2} + p_2n_j^{n-3} + \dots + p_{n-1}) \end{aligned} \quad (B18)$$

The coefficient of n^{k-1} in Equation B18 is equal to the ratio A_{kj}/A_{nj} for $k=1, \dots, n$. Hence, if, in Equation B18, we replace n^{n-k} by $x(t+(n-k)h)$ for $k=1, \dots, n$ and collect like terms in powers of n_j , then we can express $a_j \exp(\lambda_j t)$, Equation B14, as the quotient of two polynomials evaluated at n_j . We have just described the numerator of this quotient. The denominator is given by the derivative df/dn , evaluated at n_j . This completes the derivation of the algorithm given in Reference 14 (page 275).

It is clear that if $x(t)$ is a vector function and the a_k are vector constants to be determined we could apply the algorithm described above to determine the j th component of the a_k 's, for $j=1, \dots, m$. That is, an equation of the form of Equation B13 has to be solved for each component.

Recently, (References 2-4) another method has been proposed for determining the parameters occurring in Equation B1. Let us write Equation B13 as

$$\begin{bmatrix} x(t) \\ x(t+h) \\ \vdots \\ x(t+(n-1)h) \end{bmatrix} = \begin{bmatrix} a_1 & \dots & a_n \\ a_1 \eta_1 & & a_n \eta_n \\ \vdots & & \vdots \\ a_1 \eta_1^{n-1} & \dots & a_n \eta_n^{n-1} \end{bmatrix} \begin{bmatrix} \exp(\lambda_1 t) \\ \vdots \\ \exp(\lambda_n t) \end{bmatrix} \quad (\text{B19})$$

Denote the column vector on the left hand side of Equation B19 by $y(t)$, the column vector on the right hand side by $e(t)$ and the matrix by A . The Equation B19 can now be written as

$$y(t) = Ae(t) \quad (\text{B20})$$

Let \tilde{N} denote the diagonal matrix whose diagonal elements are η_1, \dots, η_n . Then

$$e(t+h) = \tilde{N}e(t) \quad (\text{B21})$$

and

$$y(t+h) = A\tilde{N}e(t) \quad (\text{B22})$$

If the matrix A is nonsingular, Equation B20 can be solved for $e(t)$ and we have

$$y(t+h) = A\tilde{N}A^{-1}y(t) \quad (\text{B23})$$

One observes the Vandermonde determinant in the determinant of the matrix in Equation B19. Hence, it is clear that A is nonsingular if the η_1, \dots, η_n all differ from one another and zero and $a_k \neq 0$ for all values of k . Note that Equation B5 and Equation B23 are equivalent statements of the same problem.

Suppose for n values of t , $t_1 < \dots < t_n$ the vectors

$$y_j = y(t_j) \quad (\text{B24})$$

and

$$z_j = y(t_j + h) \quad (B25)$$

are known. From Equations B24 and B20 we obtain the matrix equation

$$[y_1 \dots y_n] = A[e_1 \dots e_n] \quad (B26)$$

This equation can be written symbolically as

$$\phi = AE \quad (B27)$$

In a similar fashion we obtain from Equations B25 and B22

$$\psi = A\tilde{N}E \quad (B28)$$

From these two equations we obtain

$$\psi \phi^{-1}A = A\tilde{N} \quad (B29)$$

or, on setting $B = \phi^{-1}A$

$$\psi B = \phi B\tilde{N} \quad (B30)$$

Let u_k (\hat{u}_k) denote the k th column of $A(B)$ then we can replace Equations B29 and B30 by

$$\psi \phi^{-1}u_k = \eta_k u_k \quad (B31)$$

and

$$\psi \hat{u}_k = \eta_k \phi \hat{u}_k \quad (B32)$$

for $k=1, \dots, n$. Thus in the first method of this appendix the determination of the η_k 's was formulated as a problem in determining the roots of a polynomial equation, Equation B10. In this second method, the η_k 's are displayed as eigenvalues of the matrix $\psi\phi^{-1}$ or of the generalized eigenvalue problem, Equation B32. Of course, here too the determination of the η_k 's could be reduced to the determination of the roots of a polynomial equation.

Examination shows that the two methods use, or can use, exactly the same data. Since the eigenvectors are not unique we have no assurance that the first component of the vector satisfying Equation B31 is the

a_k of Equation B1. It frequently suffices to know the a_k to within a multiplicative constant. Of course, once the η_k 's are known, the matrix E is known. Hence, if E is invertible, we have from Equation B26 an equation equivalent to Equation B13 and thus the a_k 's are determined also.

In the practical situation in which one encounters Equation B1, $x(t)$ and a_k are m dimensional vectors and $n = 2m$. The Equation B1 can be written as

$$x(t) = [a_1 \dots a_n]e(t) \quad (B33)$$

Here $[a_1 \dots a_n]$, denotes the $m \times n$ matrix whose columns are the m dimensional vectors a_k , and $e(t)$ is the n dimensional column vector whose j th component is $\exp(\lambda_j t)$.

Again, setting $y(t) = x(t+h)$ we have

$$y(t) = [a_1 \dots a_n]\tilde{N}e(t) \quad (B34)$$

or

$$y(t) = [\eta_1 a_1 \dots \eta_n a_n]e(t) \quad (B35)$$

Lastly setting $z(t) = y(t+h)$, we have

$$z(t) = [\eta_1 a_1 \dots \eta_n a_n]\tilde{N}e(t) \quad (B36)$$

In block form we have from Equations B33 and B35

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a_1 & \dots & a_n \\ \eta_1 a_1 & \dots & \eta_n a_n \end{bmatrix} e(t) \quad (B37)$$

and from Equations B34 and B35

$$\begin{bmatrix} x(t+h) \\ y(t+h) \end{bmatrix} = \begin{bmatrix} a_1 & \dots & a_n \\ \eta_1 a_1 & \dots & \eta_n a_n \end{bmatrix} \tilde{N}e(t) \quad (B38)$$

Denote the matrix in Equations B37 and B38 by A. If A is nonsingular then Equation B37 can be solved for $e(t)$ and we have

$$\begin{bmatrix} x(t+h) \\ y(t+h) \end{bmatrix} = \tilde{A} \tilde{A}^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad (\text{B39})$$

Thus, the n-dimensional vector function

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad (\text{B40})$$

satisfies the system of difference relations given by Equation B39. Equation B39 is of the same form as Equation B23. Hence, as we have noted, the functions $z(t)$ satisfying Equation B39 are of the form $u \exp(\lambda t)$ where $\eta = \exp(\lambda h)$ satisfies the condition

$$\det[\tilde{A} \tilde{A}^{-1} - \eta I] = 0 \quad (\text{B41})$$

and the vector u satisfies

$$[\tilde{A} \tilde{A}^{-1} - \eta I]u = 0 \quad (\text{B42})$$

That is,

$$\tilde{A} \tilde{A}^{-1} u = \eta u \quad (\text{B43})$$

η is an eigenvalue and u is the corresponding eigenvector of the matrix $\tilde{A} \tilde{A}^{-1}$.

The matrix $\tilde{A} \tilde{A}^{-1}$ of the system Equation B39 is not known. If the vectors $x(t)$, $x(t+h)$ and $x(t+2h)$ are known for n values of t , $t_1 < \dots < t_n$, then, as above, we can determine the system matrix $\tilde{A} \tilde{A}^{-1} = \psi \phi^{-1}$ and it remains to solve the eigenvalue, eigenvector problem, either Equation B31 or Equation B32. It is clear that one immediately obtains the eigenvectors satisfying Equation B31 from those satisfying Equation B32. It is also clear that the eigenvectors so obtained are not necessarily identical to the columns of the matrix A in Equation B37.

We have described a procedure leading to an eigenvalue-eigenvector problem for the determination of the complex frequencies when $x(t)$ is a scalar and also when $x(t)$ is an m -vector and $n = 2m$. This same procedure can be used when $x(t)$ is a 2-dimensional vector, n is even and $n \geq 4$. The details are so similar to those given above that we shall not repeat them here.

Examination of the two methods of this appendix shows that they use, or can use, exactly the same data in the scalar case. When $x(t)$ is an m -dimensional vector, the first method requires $2n$ readings plus n reading for each additional component. The second method required n^2 readings to determine ϕ and n additional readings to determine ψ . Thus when $x(t)$ is an m -vector the two methods use essentially the same data also. In a similar fashion we note that it takes $3n-2$ readings to form ϕ and 2 additional readings to form ψ when $x(t)$ is a 2-dimensional vector. Exactly the same amount of data is required by the first method.

If, in Equation B1, some of the coefficients a_k are taken to be zero, then it is clear that the determinant of the matrix ϕ vanishes. Or, in other words, there are fewer a_k and λ_k to be determined than we have allowed for. Thus, the vanishing of the determinant of ϕ is a test for determining the number of terms in Equation B1. Evaluating the determinant of a matrix is usually avoided, if possible, and in numerical work a computed zero is rather rare. Hence, the vanishing of the determinant of ϕ is usually not a good test for determining the functional form of $x(t)$, Equation B1.

The process of determining the function $x(t)$, Equation B1, is analogous to interpolation with trigonometric polynomials, in the sense that the coefficients are falsified by the higher harmonics. The number of terms in Equation B1 is not specifically known. The situation may be thought of as follows. Suppose

$$x(t) = \sum_{k=1}^n a_k \exp(\lambda_k t) + z(t) \quad (\text{B44})$$

There is nothing which prevents one from forming the matrices ϕ and ψ (or determining the coefficients $c_1 \dots, c_n$, Equation B9, as described above. However, it is clear that these quantities will be falsified by the contribution of $z(t)$. Equation B44 also indicates why the vanishing of the determinant of ϕ is not a clear cut process. Taking a larger number of terms in Equation B1 might possibly reduce the influence of $z(t)$ on the parameters λ_k and a_k of Equation B1.

The columns of the matrix A defined by Equation B19 for the scalar case and by Equation B37 for the case where the a_k are m dimensional vectors, represent the eigenvectors which are to be determined. Note that these eigenvectors have a somewhat restricted form. That is, for the scalar case, the second entry in the k th column is η_k times the first entry. Similarly, for the m dimensional case the last m entries in the k th column are just the first m entries multiplied by η_k .

This special form of the eigenvectors is the basis for the modal confidence factor introduced in (Reference 5). Thus, how well the eigenvalue and eigenvector solutions of Equation B31 satisfy the special form determines the degree of confidence that the solutions so determined represent the complex modes and frequencies of the structure. The modal confidence factor can be used in addition to or as an alternative to the ratio of successive determinants test.

We conclude this section with an examination of the matrices A defined by Equation B37 and E defined by Equations B26 and B27. If $t_j = t_1 + (j-1)h$ for $j = 2, \dots, n$ then the matrix E can be written as

$$E = \begin{bmatrix} \exp \lambda_1 t & & \\ & \ddots & \\ & & \exp \lambda_n t \end{bmatrix} \begin{bmatrix} 1 & \eta_1 & \dots & \eta_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \eta_n & \dots & \eta_n^{n-1} \end{bmatrix}$$

Clearly E is nonsingular if the η_1, \dots, η_n are all different from one another. Now $\eta_k = \exp(\lambda_k h)$. Let us write $\lambda_k = \alpha_k + i\beta_k$, and suppose, for some k and $j \neq k$, $\eta_k/\eta_j = \exp h(\lambda_k - \lambda_j) = 1$.

This last equality will hold if $\alpha_k = \alpha_j$ and $|h(\beta_k - \beta_j)| = 2\pi l$. Thus even though the $\lambda_1, \dots, \lambda_m$ are all different there is the possibility of a sampling rate for which the matrix E is singular. It seems readily apparent that there should be no difficulty in choosing a sampling rate which will insure that the matrix E is nonsingular.

The columns of the matrix A, Equation B37, represent the eigenvectors associated with a linear system of first order difference equations. Accordingly the matrix A should be nonsingular. The function $x(t)$, Equation B33, perhaps more precisely, should be written as $x(t) = [c_1 a_1 \dots, c_n a_n]e(t)$. In Equation B33 the coefficients c_k are included in the symbol a_k . Thus the function $x(t)$ is a linear combination of the eigenfunctions $a_k \exp \lambda_k t$. It is clear that in a particular experiment one or more of the eigenfrequencies λ_k need not be excited. That is, those columns of A corresponding to the nonexcited λ_k are columns of zeros.

From the point of view of determining the matrix A and the eigenfrequencies λ_k it follows from our remarks that the set of sampled values of the function $x(t)$ may not be adequate for determining all the eigenvectors and frequencies. On the other hand there is the possibility that one has allowed for more eigenfrequencies in some selected frequency range than are physically present.

The difficulties noted were recognized by S. R. Ibrahim, the principal author of the method being discussed, and were treated in his papers (References 3 and 4).

APPENDIX C

SOME MATRIX EIGENVALUE-EIGENVECTOR CONSIDERATIONS

In this section we present and examine material which we feel is the basis for the method of Wittmeyer (References 6 and 7). We begin by considering the equation

$$[-\omega^2 A + B] y = r \quad (C1)$$

A and B denote symmetric matrices of order m. The matrix A is real while B is generally complex. An equation of this form represents the dynamic equation for a system with m degrees of freedom and structural damping.

The associated eigenvalue problem is

$$[-\lambda A + B] u = 0 \quad (C2)$$

We suppose there are m eigenvalues $\lambda_1, \dots, \lambda_m$ and associated eigenvectors u_1, \dots, u_m . From Equation A12 we know the eigenvectors satisfy the orthogonality conditions

$$u_j^T A u_k = 0, \quad u_j^T B u_k = 0 \quad (C3)$$

whenever $j \neq k$. Also we suppose the eigenvectors normalized so that

$$u_k^T A u_k = m_k \text{ (real)} \quad (C4)$$

then

$$u_k^T B u_k = \lambda_k m_k \quad (C5)$$

For given r and real $\omega^2 \neq \lambda_k$ for all k, Equation C1 is solvable for y. The solution y is given by the formula

$$y = \sum_{k=1}^m u_k u_k^T r / m_k (\lambda_k - \omega^2) \quad (C6)$$

It follows that

$$[-\omega^2 A + B]^{-1} = \sum_{k=1}^m u_k u_k^T / m_k (\lambda_k - \omega^2) \quad (C7)$$

We take as our first task the determination of the eigenvalues λ_k and their corresponding eigenvectors u_k . In the practical situation the known data are the values assigned to ω and the excitation vector r , and the observed responses y . However, for the present, we suppose the matrices A and B are known also. The starting point for the determination of the eigenvalues and eigenvectors is the observation that for an excitation vector $r = r_j$ satisfying the condition

$$u_k^T r_j = 0 \quad (C8)$$

for all $k \neq j$ and for any $\omega^2 \neq \lambda_k$, for all k , the solution $y = y_j$ is

$$y_j = u_j(u_j^T r_j)/m_j(\lambda_j - \omega^2) \quad (C9)$$

That is, y_j differs from the eigenvector u_j by the scalar factor $u_j^T r_j/m_j(\lambda_j - \omega^2)$.

It is rather unlikely that an r such as r_j is known. However, if λ_j is well separated from the λ_k , for $k \neq j$ and if ω^2 is "close" to λ_j , it is clear from Equation C6 that u_j is the principal contributor to the value of y . Thus for ω^2 close to λ_j we may suppose the initial guess of r for r_j expressed in the form

$$r = \sum_{k=1}^m \rho_k A u_k \quad (C10)$$

and in particular $\rho_j \neq 0$. Then by Equation C6, the corresponding y is

$$y = \sum_{k=1}^m \rho_k u_k / (\lambda_k - \omega^2) \quad (C11)$$

If the matrix A were known, we could iterate on the two operations represented by Equations C10 and C11. That is, we take

$$r^{(1)} = Ay = \sum_{k=1}^m \rho_k A u_k / (\lambda_k - \omega^2) \quad (C12)$$

then

$$y^{(1)} = \sum_{k=1}^m \rho_k u_k / (\lambda_k - \omega^2)^2 \quad (C13)$$

and so on to obtain

$$y^{(n)} = \sum_{k=1}^m \rho_k u_k / (\lambda_k - \omega^2)^{n+1} \quad (C14)$$

It is clear from Equation C14 that if for each value of n , $y^{(n)}$ is normalized then

$$y^{(n)} = y_j^{(n)} \rightarrow \text{Const. } u_j \quad (C15)$$

Thus for n sufficiently large, we will have

$$r_j^{(n)} = A y_j^{(n)} = \hat{\rho}_j A u_j$$

then from this $r_j^{(n)}$ we obtain

$$y_j^{(n+1)} = \hat{\rho}_j u_j / (\lambda_j - \omega^2) \quad (C16)$$

and from $r_j^{(n+1)} = A y_j^{(n+1)}$ we obtain

$$y_j^{(n+2)} = \hat{\rho}_j u_j / (\lambda_j - \omega^2)^2 \quad (C17)$$

We have immediately from the ratio of the v th component $y_j^{(n+1)}$ to the v th component of $y_j^{(n+2)}$; that is

$$[y_j^{(n+1)}]_v / [y_j^{(n+2)}]_v = \lambda_j - \omega^2 \quad (C18)$$

and so λ_j is known also. This is one way in which the eigenvectors u_j and the corresponding eigenvalues λ_j can be determined.

We are going to describe now yet another way for determining the eigenvector u_j and the corresponding λ_j ; A given vector y may be represented as a linear combination of the eigenvectors u_k . Thus

$$y = \sum_{k=1}^m \eta_k u_k \quad (C19)$$

Then we readily establish that

$$\begin{aligned} r &= [-\omega^2 A + B] y \\ &= \sum_{k=1}^m \eta_k (\lambda_k - \omega^2) A u_k \end{aligned} \quad (C20)$$

For this second method we start with an initial excitation vector $r^{(0)}$ and two frequencies, ω^2 and $\tilde{\omega}^2$ close to λ_j . Again supposing $r^{(0)}$ represented in the form

$$r^{(0)} = \sum \rho_k A u_k$$

we have

$$\begin{aligned} y^{(0)} &= [-\omega^2 A + B]^{-1} r^{(0)} \\ &= \sum_{k=1}^m \rho_k u_k / (\lambda_k - \omega^2) \end{aligned} \quad (C21)$$

by Equation C6. Take

$$\begin{aligned} \tilde{r}^{(0)} &= [-\tilde{\omega}^2 A + B] y^{(0)} \\ &= \sum_{k=1}^m \rho_k \frac{\lambda_k - \tilde{\omega}^2}{\lambda_k - \omega^2} a u_k \end{aligned} \quad (C22)$$

by Equation C20. Now take

$$\begin{aligned} r^{(1)} &= r^{(0)} - \tilde{r}^{(0)} \\ &= \sum_{k=1}^m \frac{\tilde{\omega}^2 - \omega^2}{\lambda_k - \omega^2} \rho_k a u_k \end{aligned} \quad (C23)$$

Considering the computations indicated by Equations C21-C23 as an iteration step, we have after $(n+1)$ steps

$$y^{(n)} = \sum_{k=1}^m \frac{(\tilde{\omega}^2 - \omega^2)^n}{(\lambda_k - \omega^2)^{n+1}} \rho_k u_k \quad (C24)$$

If the λ_k are well separated and if

$$\left| \frac{\tilde{\omega}^2 - \omega^2}{\lambda_j - \omega^2} \right| = 1 \quad (C25)$$

then obviously $y^{(n)} \rightarrow \text{const.}$ u_j and λ_j can be determined in the same fashion as above, Equation C18. In general, the equality expressed by Equation C25 is not satisfied. In this event some normalization of $y^{(n)}$ will probably be required with each iteration step. The results obtained by assuming Equation C25 held are not altered however.

In the methods discussed in this appendix up to this point we have assumed that the eigenvalues are well separated. Now we consider the case of two close eigenvalues, say λ_j and λ_{j+1} . We consider, again, a constant vector r_j satisfying the condition $u_k^T r_j = 0$ for all $k \neq j$. Thus if we excite with r_j and frequency ω_j close to λ_j we obtain

$$y_j = (u_j^T r_j) u_j / m_j (\lambda_j - \omega_j^2) \quad (C26)$$

Similarly, if we excited with the same r_j and a frequency $\hat{\omega}_j$ we obtain

$$\hat{y}_j = (u_j^T r_j) u_j / m_j (\lambda_j - \hat{\omega}_j^2) \quad (C27)$$

From Equations C26 and C27 we obtain

$$y_j - \hat{y}_j (\lambda_j - \hat{\omega}_j^2) / (\lambda_j - \omega_j^2) = y_j - \rho \hat{y}_j = 0 \quad (C28)$$

Now suppose we have linearly independent vectors r_a and r_b and that

$$r_j = a r_a + b r_b \quad (C29)$$

It follows, because of Equation C6, that the excitation vectors and frequencies r_a , ω_j , r_b , $\hat{\omega}_j$ and r_b , $\hat{\omega}_j$ will determine response vectors y_a , y_b , \hat{y}_a , and \hat{y}_b respectively. It follows also, from Equation C29, that

$$y_j = a y_a + b y_b \quad (C30)$$

$$\hat{y}_j = a \hat{y}_a + b \hat{y}_b$$

From Equation C28 we have the relation

$$\begin{aligned} y_j + \rho \hat{y}_j &= [y_a, y_b] \begin{bmatrix} a \\ b \end{bmatrix} + \rho [\hat{y}_a, \hat{y}_b] \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \left[[y_a, y_b] + \rho [\hat{y}_a, \hat{y}_b] \right] \begin{bmatrix} a \\ b \end{bmatrix} = 0 \end{aligned} \quad (C32)$$

from which we have finally

$$\left[\begin{bmatrix} r_a^T \\ r_b^T \end{bmatrix} [y_a, y_b] + \rho \begin{bmatrix} r_a^T \\ r_b^T \end{bmatrix} [\hat{y}_a, \hat{y}_b] \right] \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

APPENDIX D

DETERMINATION OF MASS, DAMPING AND STIFFNESS MATRICES

In this section we want to consider the problem of determining the mass, damping and stiffness matrices directly from experimental data. However, first, for completeness and understanding we examine the problem of solving systems of linear equations, the idea of a generalized solution and of a pseudo or generalized inverse.

Let A denote a real symmetric matrix of order m . For each such matrix A there is an associated set of m real numbers, $\lambda_1, \dots, \lambda_m$ and a linearly independent set of vectors u_1, \dots, u_m which satisfy the conditions

$$\begin{aligned} Au_k &= \lambda_k u_k \\ u_j^T A &= \lambda_j u_j^T \end{aligned} \tag{D1}$$

and

$$u_j^T u_k = \delta_{jk}$$

If we set

$$U = [u_1 \dots u_m] \tag{D2}$$

then clearly

$$U^T U = I = U U^T \tag{D3}$$

and the matrix A can be written in the form

$$A = U \Lambda U^T \tag{D4}$$

where Λ is a diagonal matrix with the λ_k 's along the main diagonal.

The properties of the matrix A as an operator are readily apparent when A is expressed in dyadic form. Thus, if we consider Ax , then we obtain from Equation D4 the following representation for the matrix A

$$A = \lambda_1 u_1 u_1^T + \dots + \lambda_m u_m u_m^T \tag{D5}$$

From Equation D5 it is clear that the range of A is the space spanned by those vectors u_k for which $\lambda_k \neq 0$. It is clear also that A is 1-1 on the range of A. Hence, if the range and domain of A are the same, that is, if $\lambda_k \neq 0$ for all k, then A^{-1} exists and

$$A^{-1} = (1/\lambda_1)u_1u_1^T + \dots + (1/\lambda_m)u_mu_m^T \quad (D6)$$

Let N_1 denote the set of values of k for which $\lambda_k \neq 0$ and N_0 the set of values of k for which $\lambda_k = 0$. From our remarks above, $Ax = b$ has a "strict" solution if and only if b lies in the range of A. Now b lies in the range of A if and only if

$$u_k^T b = 0 \quad (D7)$$

for all k in N_0 . Thus if b satisfies these solvability conditions then a solution to $Ax = b$ is given by

$$x \equiv \sum_{k \in N_1} (1/\lambda_k)u_ku_k^T b \quad (D8)$$

If we set

$$y = x + \sum_{k \in N_0} c_k u_k \quad (D9)$$

then $Ay = b$ also and

$$||y|| \geq ||x|| \quad (D10)$$

Even if b does not satisfy the solvability conditions, we can still compute an x by means of Equation D8. Then

$$Ax = \sum_{k \in N_1} u_k u_k^T b \quad (D11)$$

and

$$b - Ax = \sum_{k \in N_0} u_k u_k^T b \quad (D12)$$

Then for any y whatsoever

$$b - Ay = \sum_{k \in N_1} c_k u_k + \sum_{k \in N_0} u_k u_k^T b \quad (D13)$$

It follows from Equation D12 and D13 that

$$||b - Ay|| \geq ||b - Ax|| \quad (D14)$$

Set

$$A^I = \sum_{k \in N_1} (1/\lambda_k) u_k u_k^T \quad (D15)$$

A^I is called the generalized or pseudo inverse of A . We have seen that if λ_k is not zero for any value of k then $A^I = A^{-1}$. If the equation $Ax = b$ has solutions then $x = A^I b$ is the solution of least magnitude, Equation D10. On the other hand, if $Ax = b$ has no solutions then $x = A^I b$ satisfies the equation $Ax = b$ as well as or better than any other vector y (Equation D14).

We have characterized the generalized inverse for a real symmetric matrix A . Next we want to characterize the generalized inverse of an arbitrary real $(m \times n)$ matrix A , where $m \geq n$, and of its transpose A^T . To achieve our purpose we consider the matrix $A^T A$.

Now the matrix $A^T A$ is real symmetric and of order n . Hence there are real eigenvalues $\epsilon_1, \dots, \epsilon_n$ and a complete orthonormal set of n dimensional eigenvectors u_1, \dots, u_n . It follows from the equation

$$u_k^T A^T A u_k = \epsilon_k u_k^T u_k = \epsilon_k \quad (D16)$$

that $\epsilon_k \geq 0$. Set $\lambda_k = \sqrt{\epsilon_k} \geq 0$ and set

$$v_k = (1/\lambda_k) A u_k, \text{ if } \lambda_k \neq 0 \quad (D17)$$

$$v_k = A u_k = 0$$

if $\lambda_k = 0$. One readily shows that the v_k , for k in N_1 , constitute a set of orthonormal m - dimensional vectors.

We obtain from Equation D17, regardless of the value of λ_k ,

$$A[u_1, \dots, u_n] = [v_1, \dots, v_n]\Lambda \quad (D18)$$

From this equation we infer that A has a representation

$$A = \sum_{k \in N_1} \lambda_k v_k u_k^T \quad (D19)$$

then, as above, we take

$$A^T = \sum_{k \in N_1} (1/\lambda_k) u_k v_k^T \quad (D20)$$

We have from Equation D19 that

$$A^T = \sum_{k \in N_1} \lambda_k u_k v_k^T \quad (D21)$$

and hence

$$(A^T)^T = \sum_{k \in N_1} (1/\lambda_k) v_k u_k^T \quad (D22)$$

In a manner similar to the above one can obtain results of the same nature for complex matrices.

We now describe a rather simple and obvious approach for determining the mass, damping and stiffness matrices. In general this procedure fails in practical situations and hence is really no method.

The steady state response of the system of equations $M\ddot{x} + C\dot{x} + Kx = f$ to a harmonic excitation $f(t) = r \exp(i\omega_1 t)$, r a constant vector, is of the form $x(t) = y \exp(i\omega_1 t)$ where y is a constant vector. If for $k = 1, \dots, m$ we take r equal to the unit Cartesian coordinate vectors respectively we obtain the matrix equation

$$[-\omega_1^2 M + K + i\omega_1 C] [y_1, \dots, y_m] = I \quad (D23)$$

Since ω_1 is not a characteristic value we can infer that the matrix $[y_1, \dots, y_m]$ is nonsingular. It follows then from Equation D23 that

$$-\omega_1^2 M + K = \text{Re}[y_1, \dots, y_m]^{-1} \quad (\text{D24})$$

and

$$\omega_1 C = \text{Im}[y_1, \dots, y_m]^{-1} \quad (\text{D25})$$

If we repeated the process for $\omega = \omega_2$ then we would have another equation of the form of Equation D24, say

$$-\omega_2^2 M + K = \text{Re}[z_1, \dots, z_m]^{-1} \quad (\text{D26})$$

The Equation D25 gives the damping matrix C and Equations D24 and D26 determine the mass and stiffness matrices. It comes as somewhat of a surprise that the M , C and K determined by this method are not very good. Of course, one immediately concludes that the matrices $[y_1, \dots, y_m]$ and $[z_1, \dots, z_m]$ must be ill conditioned. If one looks at Equation A34 or Equation A38 of Appendix A the reason for the ill conditioning becomes apparent. Thus ω_1 will be close to λ_j say and for $k = 1, \dots, m$ the y_k will have nearly the same direction, as the modal vector u_j . This observation coupled with the limited accuracy of experimental data explains the poor results obtained by this method.

It is clear that one can obtain a better set of spanning vectors $[y_1, \dots, y_m]$ by a judicious choice of frequencies, $\omega_1, \dots, \omega_m$. However, the resulting set of equations are more complicated. This method was discussed recently in Reference 8 and we want to examine the method here.

In Reference 8 it was assumed that the mass matrix M was known. It is instructive to consider first that all three matrices M , C and K are unknown.

Let $y_k \exp(i\omega_k t)$ denote the steady state response to the harmonic excitation $r_k \exp(i\omega_k t)$, r_k real. We have then the equation

$$-M\omega_k^2 y_k + Ky_k + iC\omega_k y_k = r_k \quad (D27)$$

Set $y_k = y_{Rk} + iy_{Ik}$. Then Equation D27 can be rewritten as two real equations

$$\begin{aligned} M(-\omega_k^2 y_{Ik}) + Ky_{Rk} + C(-\omega_k y_{Ik}) &= r_k \\ M(-\omega_k^2 y_{Ik}) + Ky_{Ik} + C(\omega_k y_{Rk}) &= 0 \end{aligned} \quad (D28)$$

Now let us suppose m , the order of the matrices M , C and K is even and set $p = 3m/2$. Also, set

$$\begin{aligned} \hat{y}_k^T &= [-\omega_k^2 y_{Rk}^T, y_{Rk}^T, -\omega_k y_{Ik}^T] \\ \hat{y}_{p+k}^T &= [-\omega_k^2 y_{Ik}^T, y_{Ik}^T, \omega_k y_{Rk}^T] \end{aligned} \quad (D29)$$

For $k=1, \dots, p$ we have then the system of equations

$$[M \ K \ C][\hat{y}_1, \dots, \hat{y}_p, \hat{y}_{p+1}, \dots, \hat{y}_{2p}] = [r_1, \dots, r_p, 0 \dots 0] \quad (D30)$$

where $[M \ K \ C]$ is an $m \times 3m$ matrix, $[\hat{y}_1 \dots \hat{y}_{2p}]$ is $3m \times 3m$ and $[r_1 \dots r_p, 0 \dots 0]$ is $m \times 3m$. In view of our remarks above on generalized inverses we may write

$$[M \ K \ C] = [r_1 \dots r_p, 0 \dots 0][\hat{y}_1 \dots \hat{y}_{2p}]^{-1}$$

where if $[\hat{y}_1 \dots \hat{y}_{2p}]$ is not strictly invertible, then $[\hat{y}_1 \dots \hat{y}_{2p}]^{-1}$ denotes the appropriate generalized inverse of $[\hat{y}_1 \dots \hat{y}_{2p}]$.

Let us observe that if any row of the matrix $[r_1 \dots r_p, 0 \dots 0]$ consists entirely of zeros then the same row of the matrix $[M \ K \ C]$ will have only zeros. It follows that the r_1, \dots, r_p should span an m dimensional space. Alternatively, it follows that the system determined by the matrices M , C and K should be excited at least once at each station, if we are to avoid a row of zeros in the matrices M , C and K .

For the case where the matrices M , C and K are of order m (i.e. for a true m degree of freedom system) it is possible to choose the amplitudes r_k and the frequencies ω_k so that the matrix $[\hat{y}_1, \dots, \hat{y}_{2m}]$ has a strict inverse. That is, in principle it is possible to recover the matrices M , C and K from experiments. For the practical case, however, many of the points at which one observes the response to an excitation are not suitable points for exciting the structure.

For the reasons just mentioned determining the matrices M , C and K by the method represented by Equation D31 has very limited applicability. The situation is quite different if the mass matrix M is known. In this case set

$$\begin{aligned}\hat{y}_k^T &= [y_{Rk}^T, -\omega_k y_{Ik}^T] \\ \hat{y}_{m+k}^T &= [y_{Ik}^T, \omega_k y_{Rk}^T]\end{aligned}\tag{D32}$$

and

$$\begin{aligned}\hat{r}_k &= r_k + \omega_k^2 M y_{Rk} \\ \hat{r}_{m+k} &= \omega_k^2 M y_{Ik}\end{aligned}$$

Then for $k=1, \dots, m$ we have the system of equations for the $m \times 2m$ matrix $[K \ C]$

$$[K \ C][\hat{y}_1, \dots, \hat{y}_{2m}] = [\hat{r}_1, \dots, \hat{r}_{2m}]\tag{D33}$$

The vector r_k in the expression $r_k + \omega_k^2 M y_{Rk}$ can be the same vector for all values of k . That is, if the matrix M is known then exciting the system at a single well chosen station with a set of well chosen frequencies $\omega_1, \dots, \omega_m$ the matrix $[\hat{y}_1, \dots, \hat{y}_{2m}]$ will be strictly invertible when the eigenvalues λ_k are well separated.

If for some value of k , λ_k is an eigenvalue of multiplicity greater than 1 then it is clear from Equation A38 that the matrix $[\hat{y}_1, \dots, \hat{y}_{2m}]$, obtained by exciting the system at a single point, will be singular.

Suppose the eigenvalue λ_k is of multiplicity p and for simplicity suppose also that λ_k is the only multiple eigenvalue. In this case we need p suitable points at which to excite the system. (Even though p excitation points are required, the system is excited at only one point at a time.) Then for near resonance frequencies $\omega_1, \dots, \omega_{m-p+1}$ we should, at least in principle, obtain a matrix $[\hat{y}_1, \dots, \hat{y}_{2m}]$ which is strictly invertible. For the frequency ω_k which is close to the multiple eigenvalue λ_k the system should be excited at each of the p excitation points, one point at a time.

Thus corresponding to each multiple eigenvalue we need a set of excitation points equal in number to the multiplicity of the eigenvalue. The same excitation points, if suitable, may be used for different eigenvalues. We believe a similar procedure should be followed for close eigenvalues also.

Set $Y_R = [Y_{R1}, \dots, Y_{RM}]$, $Y_I = [Y_{I1}, \dots, Y_{IM}]$, $R = [r_1, \dots, r_m]$ and let Ω denote the diagonal matrix with diagonal elements ω_k for $k=1, \dots, m$. Then Equation D33 can be written in block matrix form as

$$\begin{aligned} KY_R - CY_I\Omega &= R + MY_R\Omega^2 \\ KY_I + CY_R\Omega &= MY_I\Omega^2 \end{aligned} \tag{D34}$$

If we multiply Equations D34 (on the left) by M^{-1} we obtain a system of equations for the transformed stiffness matrix $K^* = M^{-1}K$ and damping matrix $C^* = M^{-1}C$, namely

$$\begin{aligned} K^*Y_R - C^*Y_I\Omega &= R + Y_R\Omega^2 \\ K^*Y_I + C^*Y_R\Omega &= Y_I\Omega^2 \end{aligned} \tag{D35}$$

If, for example, we multiply the second of the Equations D35 on the right by $Y_I^{-1}Y_R$ and subtract the first equation therefrom we obtain

$$C^*[Y_R\Omega Y_I^{-1}Y_R + Y_I\Omega] = Y_I\Omega^2 - R^* - Y_R\Omega^2 \quad (D36)$$

Once this equation is solved for C^* the value so obtained can be used in either of the Equations D35 to determine K^* . A procedure of this kind is probably preferable for determining the matrices K and C or their transforms K^* and C^* , to solving Equation D33 for the matrix $[K \ C]$, for example.

We have examined the method (Reference 8) primarily from the point of view of an evenly determined system. If the systems Equations D33, Equation D34 or Equations D35 were over or under determined then the methods described at the beginning of this appendix could be used to determine matrices $K(K^*)$ and $C(C^*)$. It should be clear from our remarks above that even though the number of experiments exceeds the number of unknowns we do not have, necessarily, an over determined system. Once the matrices $K(K^*)$ and $C(C^*)$ are determined one can calculate other quantities of interest as described in Reference 8.

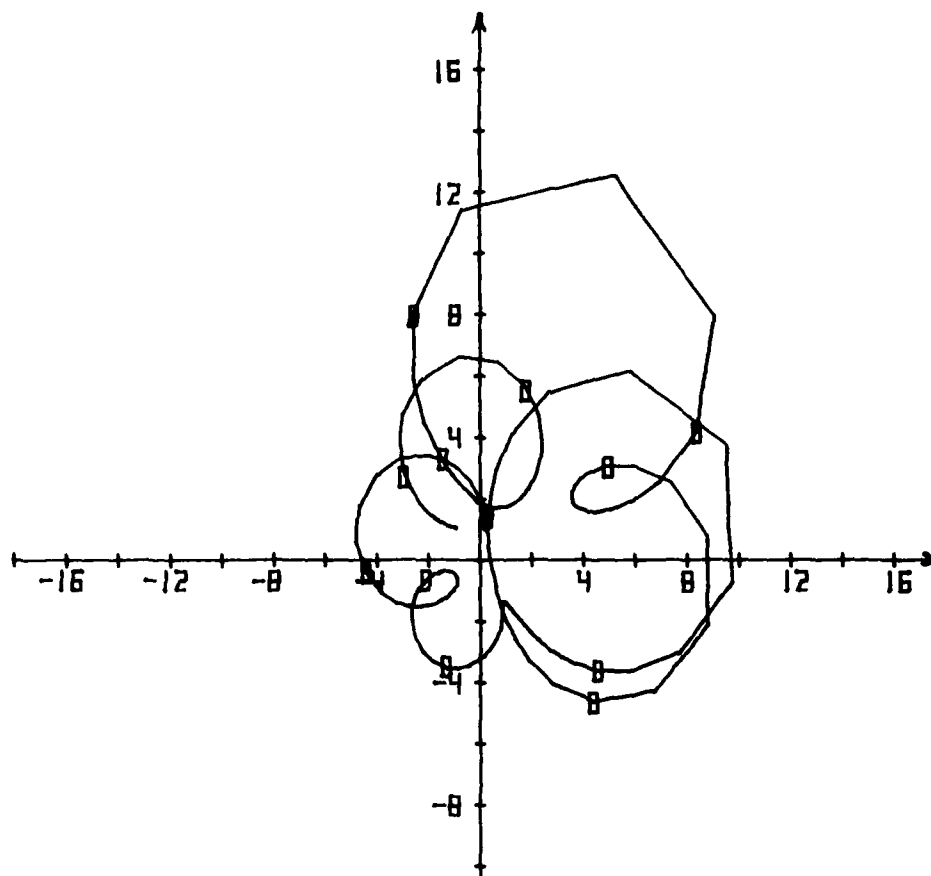


Figure 1. An Example of an Element of the Frequency Response Function Matrix

Graph in a complex plane of the function

$$w(\omega) = \xi(\omega) + i\eta(\omega) = \sum_{k=1}^6 [b_k/(i\omega - \lambda_k) + \bar{b}_k/(i\omega - \bar{\lambda}_k)]$$

The characteristic values λ_k and the coefficients b_k are given in Table 1A. The boxes are the values of $w(\omega)$ for the frequencies listed in Table 1A.

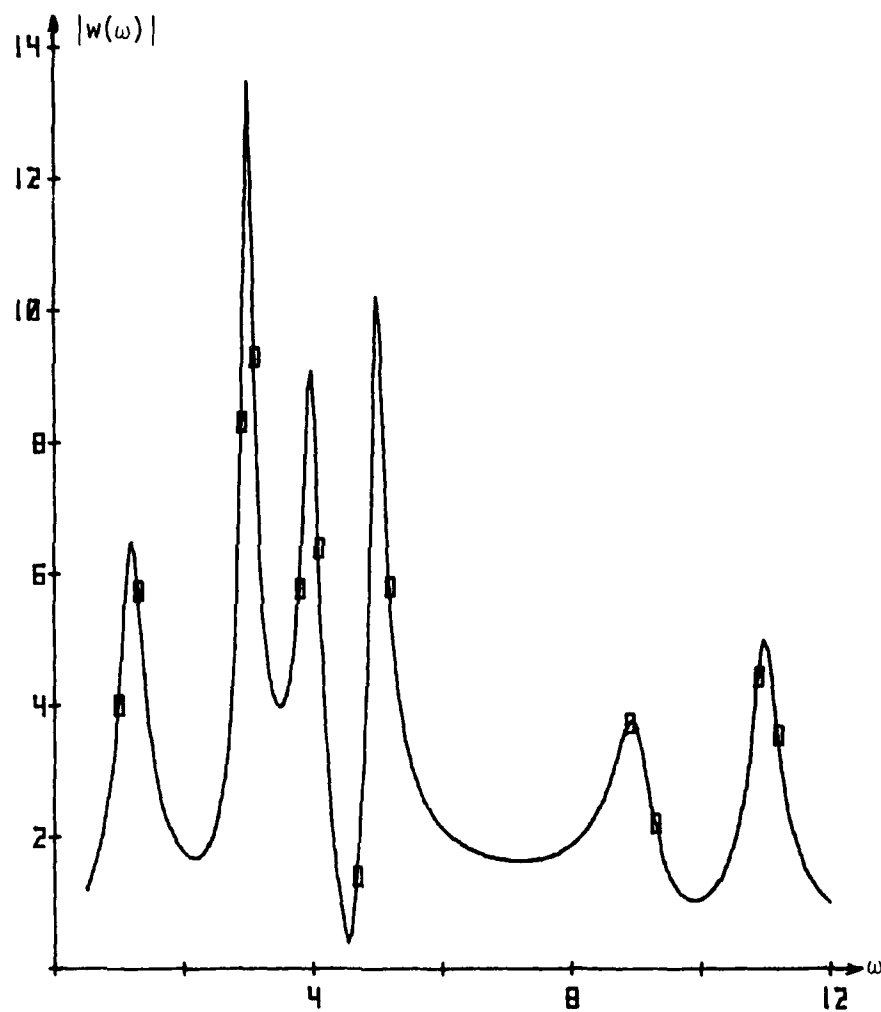


Figure 2. Graph of the Magnitude of $w(\omega)$

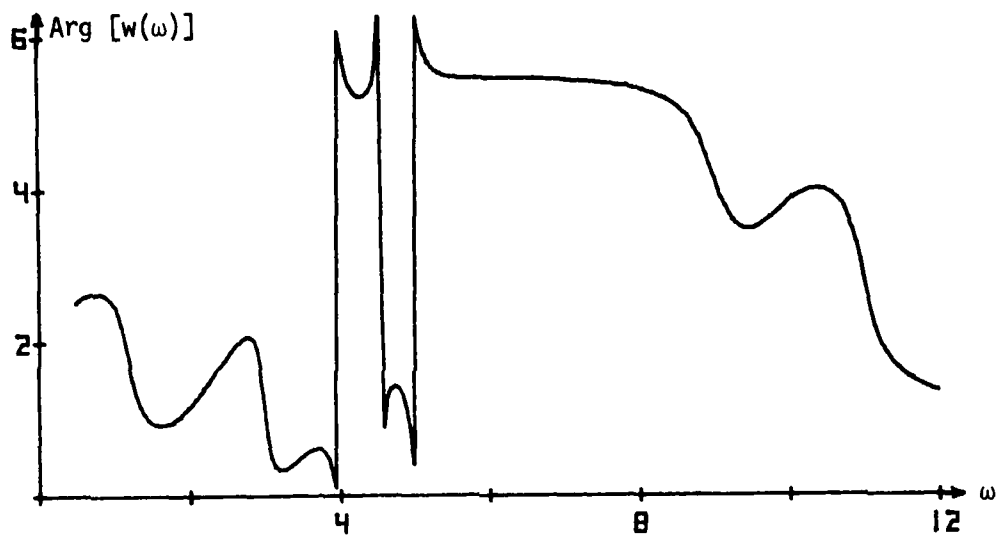


Figure 3. Graph of the Argument of $w(\omega)$

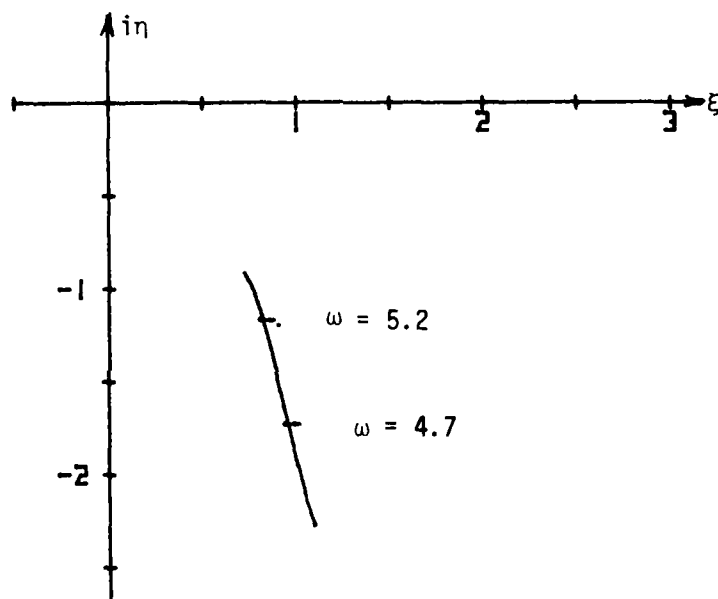


Figure 4. Graph in the Complex Plane of $\xi(\omega) + i\eta(\omega) = w(\omega) - b_4 / (i\omega - \lambda_4)$

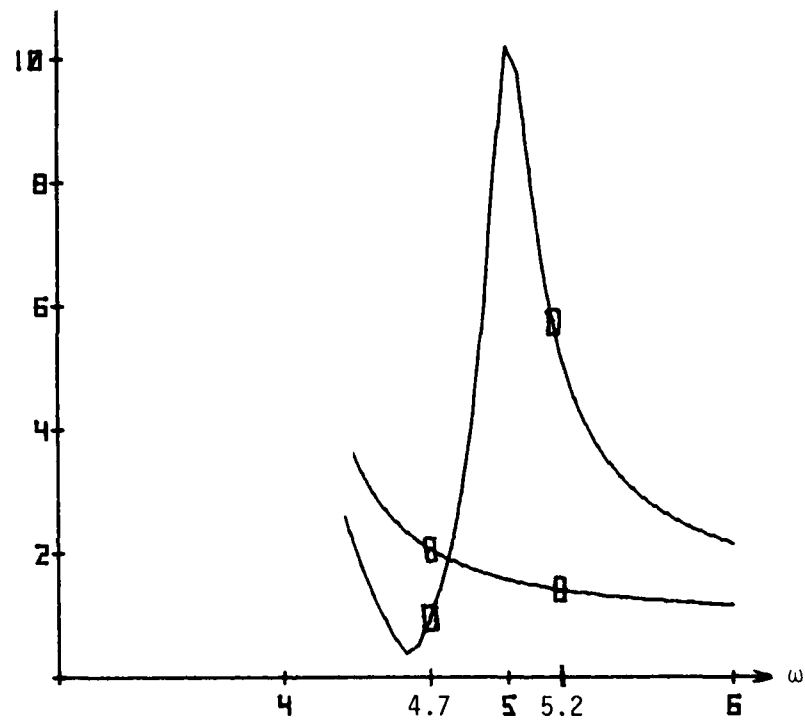
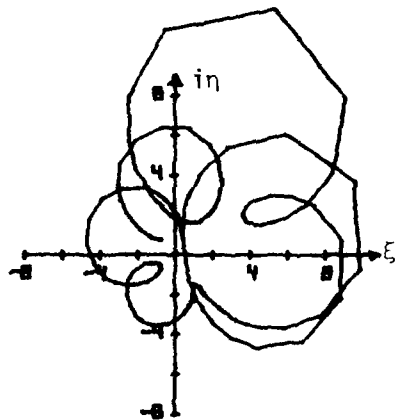
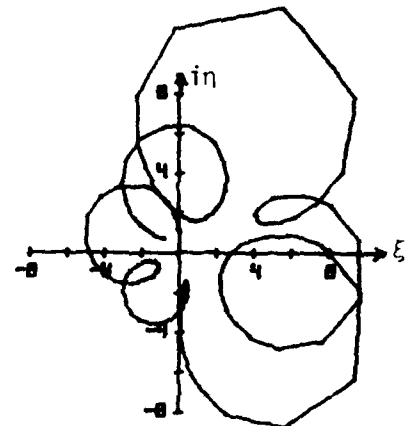


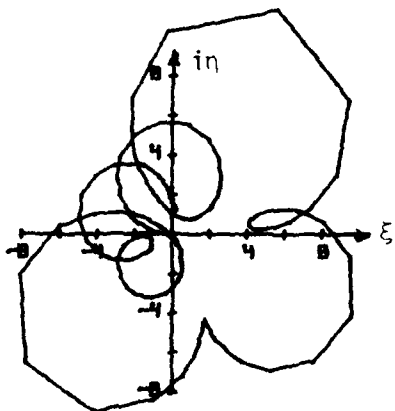
Figure 5. "Upper Curve" - Graph of $|w(\omega)|$ "Lower Curve" - Graph of $|w(\omega) - b_4/(i\omega - \lambda_4)|$



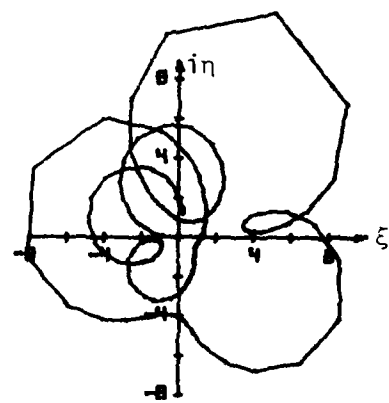
$$b_4 = 0.866025404 + 0.5i$$



$$b_4 = 0.866025404 - 0.5i$$

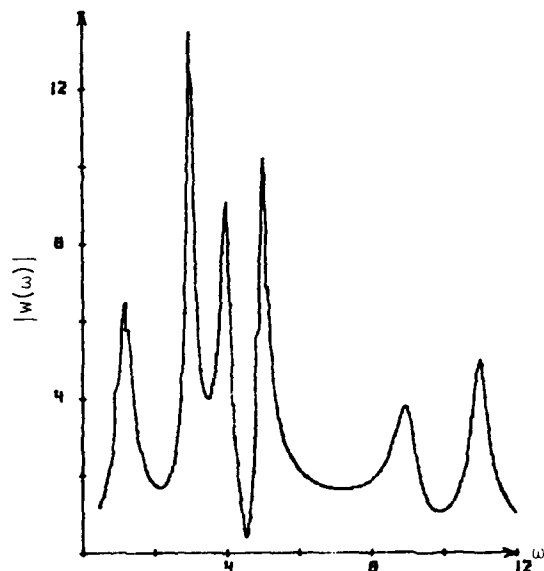


$$b_4 = -0.866025404 - 0.5i$$

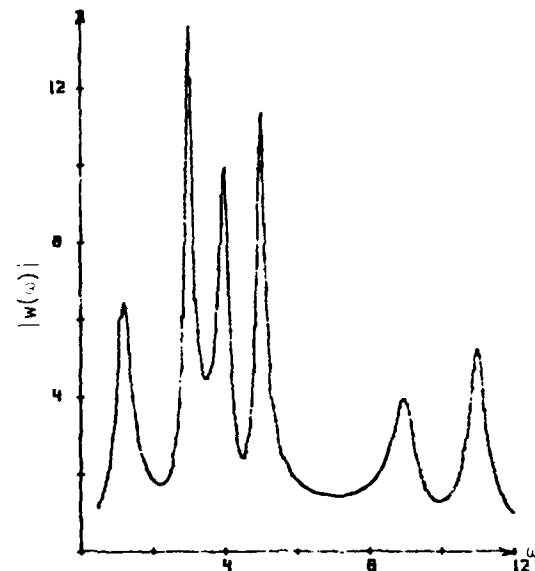


$$b_4 = -0.866025404 + 0.5i$$

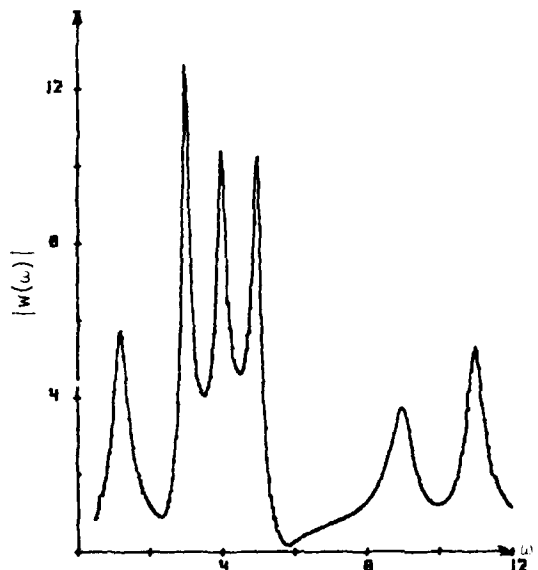
Figure 6a. Graphs of $w(\omega)$ for Four Values of b_4



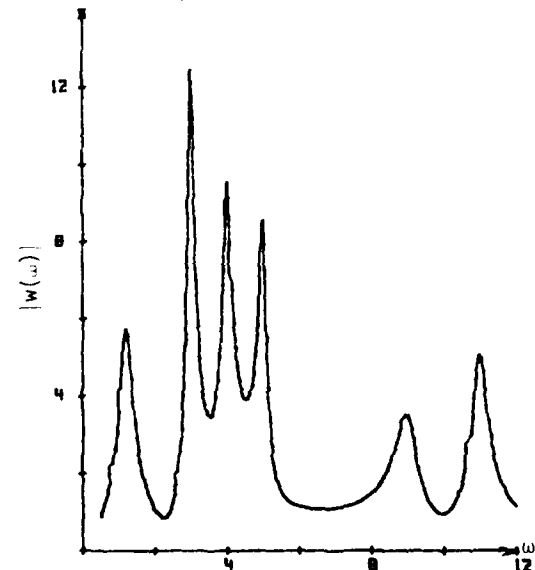
$$b_4 = 0.866025404 + 0.5i$$



$$b_4 = 0.866025404 - 0.5i$$



$$b_4 = -0.866025404 - 0.5i$$



$$b_4 = -0.866025404 + 0.5i$$

Figure 6b. Graphs of $|w(\omega)|$ for the Functions $w(\omega)$ of Figure 6a

TABLE 1A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$b_4 = 0.866025404 + 0.5i$$

CHARACTERISTIC VALUES

λ_k			
-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	5.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01

COEFFICIENTS

b_k			
-1.73648178E-01	9.84807753E-01	3.42820143E-01	9.39692621E-01
9.39692621E-01	-3.42820143E-01	8.66825404E-01	5.00000000E-01
-7.66844443E-01	-6.42787610E-01	-9.39692621E-01	3.42820143E-01

FREQUENCIES ω_p

1.00000000E+00
2.90000000E+00
3.80000000E+00
4.70000000E+00
8.90000000E+00
1.09000000E+01

 ω_{6+p}

1.30000000E+00
3.10000000E+00
4.10000000E+00
5.20000000E+00
9.30000000E+00
1.12000000E+01

FUNCTION VALUES

-4.29273852E+00
-3.34913691E+00
4.38941589E+00
-2.33935099E-01
-1.66995819E+00
-4.71525428E+00
5.99721924E-01
7.61406142E+00
3.77740465E+00
4.83893134E+00
-2.44041139E+00
-1.71451304E+00

2.25175077E+00
7.62776327E+00
2.75193313E+00
1.16599791E+00
-3.62327685E+00
-5.22320098E-01
5.07376883E+00
3.84211833E+00
-4.89851862E+00
-3.82058736E+00
-7.77992528E-01
3.16962769E+00

TRUNCATED VALUES

-4.29000000E+00
-3.34000000E+00
4.38000000E+00
-2.30000000E-01
-1.66000000E+00
-4.71000000E+00
5.98000000E-01
7.61000000E+00
3.77000000E+00
4.83000000E+00
-2.44000000E+00
-1.71000000E+00
2.25000000E+00
7.62000000E+00
2.75000000E+00
1.16000000E+00
-3.62000000E+00
-5.20000000E-01
5.07000000E+00
3.84000000E+00
-4.89000000E+00
-3.82000000E+00
-7.78000000E-01
3.16000000E+00

TABLE 1B
COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 3.641182648E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.749999998E-01	1.200000005E+00	-8.500000002E-02	3.000000003E+00
-1.250000040E-01	3.999999998E+00	-1.000000038E-01	4.999999996E+00
-2.999999979E-01	9.000000004E+00	-2.000000012E-01	1.000000000E+01

CHARACTERISTIC VALUE DIFFERENCES

-1.035000000E-09	-5.200000000E-09	2.290000000E-11	-2.630000000E-09
4.003000000E-09	1.960000000E-09	3.829000000E-09	4.490000000E-09
-2.090000000E-09	-3.640000000E-09	1.226000000E-09	-4.000000000E-10

COMPUTED COEFFICIENTS

b_k			
-1.736481367E-01	9.848077539E-01	3.420201700E-01	9.396926135E-01
9.396926567E-01	-3.420201388E-01	8.660254162E-01	5.000000180E-01
-7.660444440E-01	-6.427875893E-01	-9.396926269E-01	3.420201491E-01

COEFFICIENT DIFFERENCES

-4.129100000E-08	-8.500000000E-10	-2.702200000E-08	7.524000000E-09
-3.568000000E-08	-4.219000000E-09	-1.224000000E-08	-1.002100000E-08
9.770000000E-10	-2.071700000E-08	5.066000000E-09	-6.009000000E-09

TABLE 1C

COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 3. 620600949E-08

COMPUTED CHARACTERISTIC VALUES

-1. 752927073E-01	λ_k 1. 200003610E+00	-8. 507245055E-02	3. 000099041E+00
-1. 248179243E-01	3. 999900107E+00	-9. 981410287E-02	5. 000273706E+00
-2. 908711876E-01	9. 001500001E+00	-1. 999952024E-01	1. 009947000E+01

CHARACTERISTIC VALUE DIFFERENCES

2. 927072570E-04	-3. 610310000E-06	7. 245855050E-05	-9. 904055000E-05
-1. 820757100E-04	9. 909276000E-05	-1. 850971200E-04	-2. 737063000E-04
-1. 120012368E-03	-1. 500000620E-03	-4. 797560000E-06	5. 219153000E-04

COMPUTED COEFFICIENTS

-1. 744831047E-01	b_k 9. 045283279E-01	3. 426584037E-01	9. 389229542E-01
9. 375117004E-01	-3. 402454944E-01	8. 642405451E-01	4. 974561162E-01
-7. 652318548E-01	-6. 376405929E-01	-9. 306066391E-01	3. 396870933E-01

COEFFICIENT DIFFERENCES

8. 349266750E-04	2. 794250940E-04	-6. 302607320E-04	7. 696660020E-04
2. 100920504E-03	-1. 774640649E-03	1. 704050000E-03	2. 543083013E-03
-8. 125001640E-04	-5. 147017073E-03	-1. 005901071E-03	2. 413049720E-03

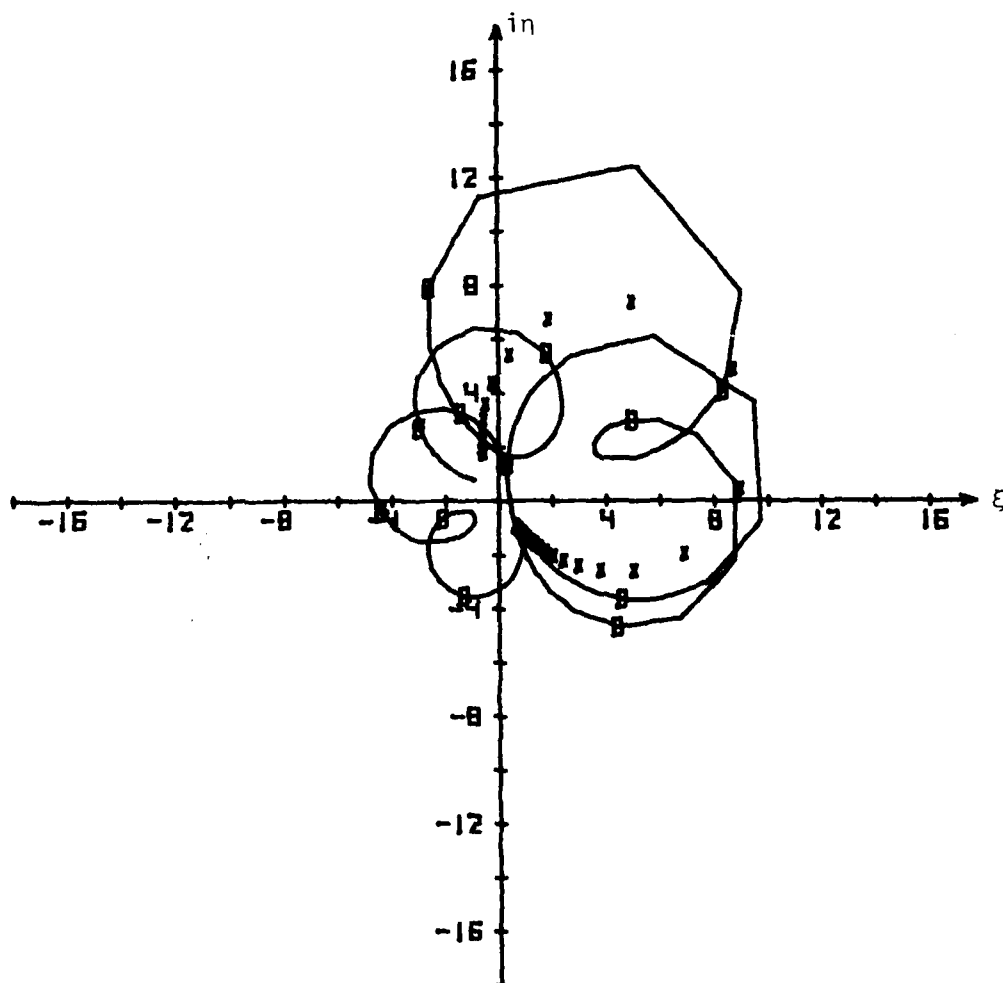


Figure 7a. Graph of $w(\omega)$, Coefficient $b_4 = 0.866025404 + 0.5i$

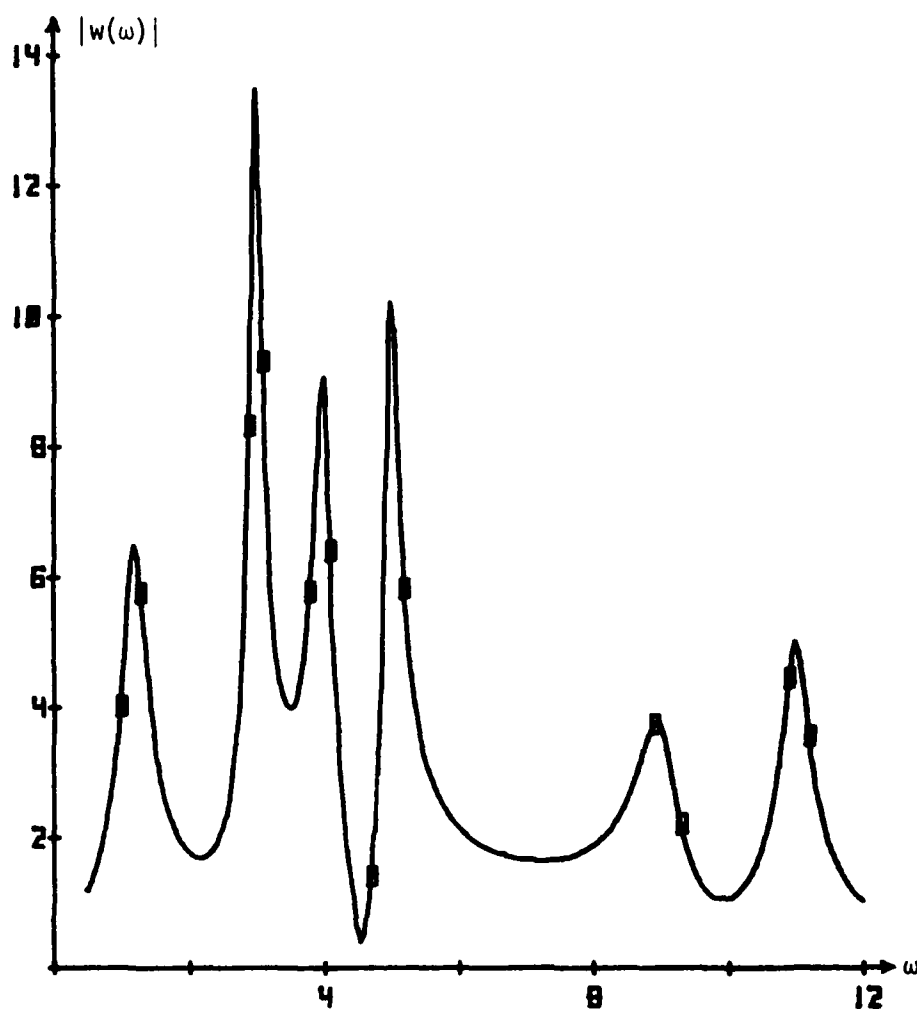


Figure 7b. Graph of $|w(\omega)|$, Coefficient $b_4 = 0.866025404 + 0.5i$

TABLE 2A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$b_4 = 0.866025404 - 0.5i$$

CHARACTERISTIC VALUES

 λ_k

-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	5.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01

COEFFICIENTS

 b_k

-1.73648178E-01	9.04087753E-01	3.42020143E-01	9.39692621E-01
9.39692621E-01	-3.42020143E-01	8.66825404E-01	-5.00000000E-01
-7.66844443E-01	-6.42787610E-01	-9.39692621E-01	3.42020143E-01

FREQUENCIES

 ω_p ω_{6+p}

1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
3.80000000E+00	4.10000000E+00
4.70000000E+00	5.20000000E+00
8.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01

FUNCTION VALUES

TRUNCATED VALUES

-3.87627428E+00	2.24828168E+00	-3.87000000E+00	2.24000000E+00
-2.74746179E+00	7.60674089E+00	-2.74000000E+00	7.60000000E+00
5.25062379E+00	2.60425876E+00	5.25000000E+00	2.60000000E+00
2.86914672E+00	1.67060616E-01	2.86000000E+00	1.60000000E-01
-1.85426125E+00	-3.62932881E+00	-1.85000000E+00	-3.62000000E+00
-4.82180654E+00	-5.24797184E-01	-4.82000000E+00	-5.20000000E-01
1.02848589E+00	5.06898845E+00	1.02000000E+00	5.06000000E+00
8.26236120E+00	3.81681795E+00	8.26000000E+00	3.81000000E+00
4.90484246E+00	-5.01926240E+00	4.90000000E+00	-5.01000000E+00
1.36961136E-01	-5.81962628E+00	1.30000000E-01	-5.81000000E+00
-2.60291717E+00	-7.82900936E-01	-2.60000000E+00	-7.80000000E-01
-1.81403537E+00	3.16740794E+00	-1.81000000E+00	3.16000000E+00

TABLE 2B
COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 11

RADIUS OF DIFFERENCES 4. 009709056E-08

COMPUTED CHARACTERISTIC VALUES

-1. 749999904E-01	λ_k 1. 200000006E+00	-8. 500000028E-02	3. 000000003E+00
-1. 250000044E-01	3. 999999997E+00	-9. 999999700E-02	4. 999999994E+00
-2. 999999904E-01	9. 000000004E+00	-2. 000000013E-01	1. 100000000E+01

CHARACTERISTIC VALUE DIFFERENCES

-1. 619000000E-09	-5. 530000000E-09	2. 792000000E-10	-3. 190000000E-09
4. 434000000E-09	2. 700000000E-09	-2. 997000000E-09	5. 960000000E-09
-1. 602000000E-09	-3. 960000000E-09	1. 311000000E-09	-6. 000000000E-10

COMPUTED COEFFICIENTS

-1. 736481333E-01	b_k 9. 040077510E-01	3. 420201762E-01	9. 396926146E-01
9. 396926618E-01	-3. 420201343E-01	8. 660254154E-01	-4. 999999783E-01
-7. 660444465E-01	-6. 427875892E-01	-9. 396926273E-01	3. 420201503E-01

COEFFICIENT DIFFERENCES

-4. 465900000E-08	1. 971000000E-09	-3. 318500000E-08	6. 441000000E-09
-4. 001000000E-08	-8. 717000000E-09	-1. 130000000E-08	-2. 166600000E-08
3. 549000000E-09	-2. 070600000E-08	6. 317000000E-09	-7. 260000000E-09

TABLE 2C

COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 11

RADIUS OF DIFFERENCES 4. 112217381E-08

COMPUTED CHARACTERISTIC VALUES

-1. 749889211E-01	1. 199993105E+00	-8. 499958878E-02	3. 000894968E+00
-1. 252792343E-01	3. 999796988E+00	-9. 978799277E-02	5. 000177948E+00
-3. 001535183E-01	9. 000694848E+00	-2. 000615370E-01	1. 099947454E+01

CHARACTERISTIC VALUE DIFFERENCES

-1. 907885100E-05	6. 894740000E-06	-4. 112157000E-07	-9. 496827000E-05
2. 792343460E-04	2. 010924600E-04	-2. 920072310E-04	-1. 779475100E-04
1. 535103240E-04	-6. 948403600E-04	6. 153690600E-05	5. 254552000E-04

COMPUTED COEFFICIENTS

-1. 744182864E-01	9. 821542212E-01	3. 428315832E-01	9. 385152680E-01
9. 402540961E-01	-3. 41368353E-01	8. 627195752E-01	-4. 989461049E-01
-7. 662658965E-01	-6. 398903351E-01	-9. 392122933E-01	3. 403523720E-01

COEFFICIENT DIFFERENCES

7. 621083790E-04	2. 653531781E-03	-8. 114401530E-04	1. 177352994E-03
-5. 614751220E-04	-6. 833076510E-04	3. 385828779E-03	-1. 053815878E-03
2. 214534740E-04	-2. 897274887E-03	-4. 803276640E-04	1. 667770906E-03

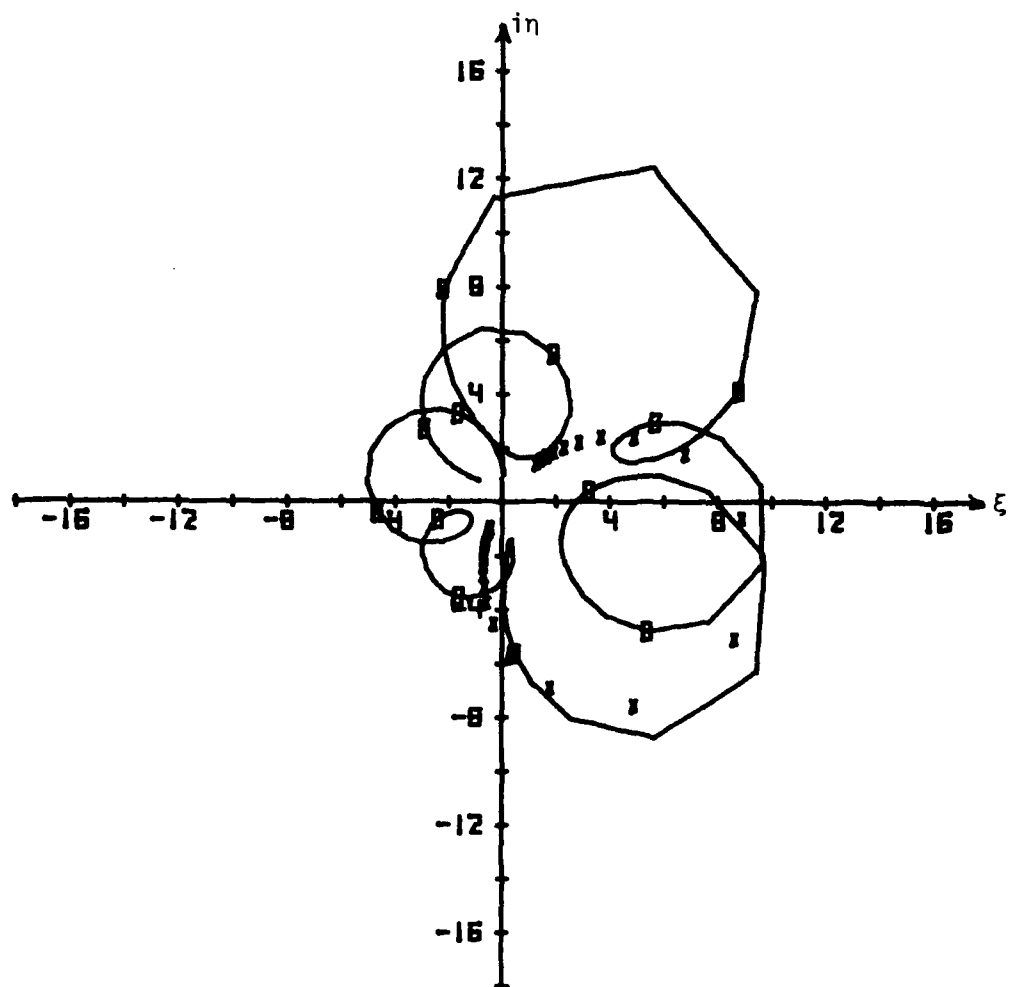


Figure 8a. Graph of $w(\omega)$, Coefficient $b_4 = 0.866025404 - 0.5i$

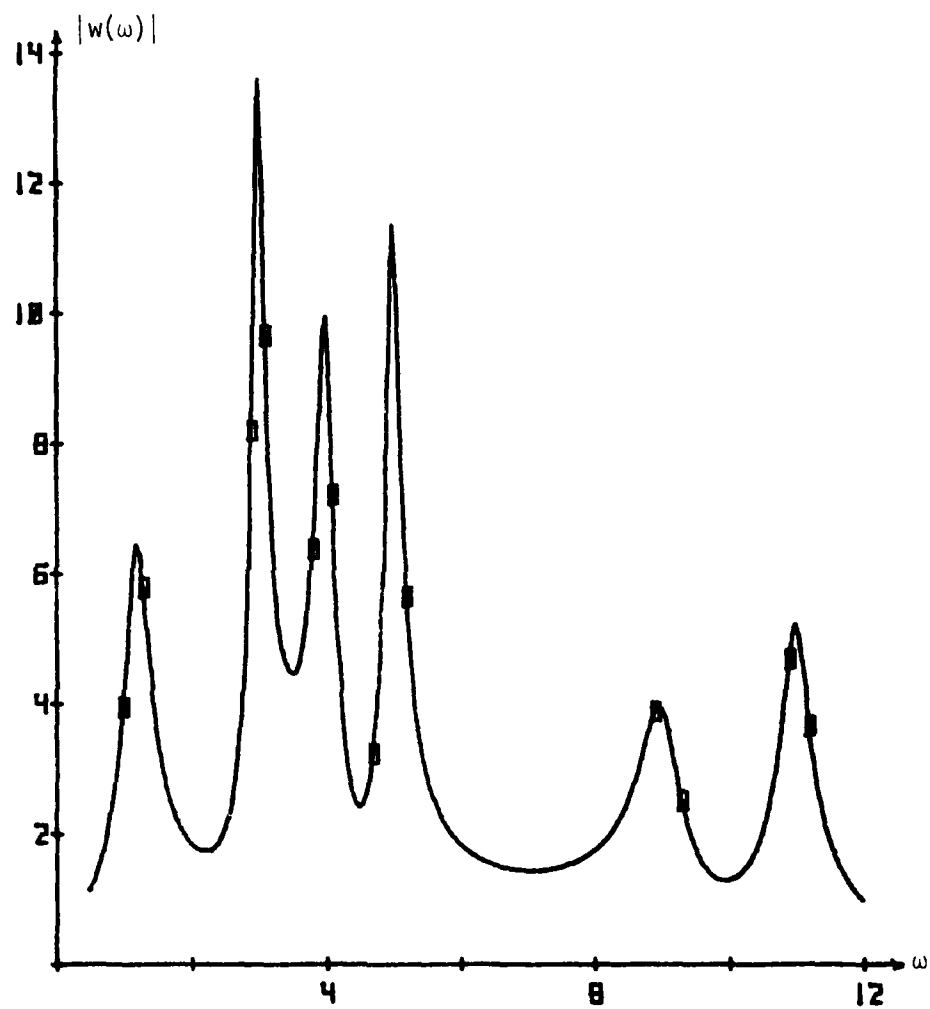


Figure 8b. Graph of $|w(\omega)|$, Coefficient $b_4 = 0.866025404 - 0.5i$

TABLE 3A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$b_4 = -0.866025404 - 0.5i$$

CHARACTERISTIC VALUES

 λ_k

-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	5.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01

COEFFICIENTS

 b_k

-1.73648170E-01	9.04807753E-01	3.42820143E-01	9.39692621E-01
9.39692621E-01	-3.42820143E-01	-8.66825404E-01	-5.00000000E-01
-7.66044430E-01	-6.42787610E-01	-9.39692621E-01	3.42820143E-01

FREQUENCIES

 ω_p ω_{p+6}

1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
3.80000000E+00	4.10000000E+00
4.70000000E+00	5.20000000E+00
8.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01

FUNCTION VALUES

-3.891902761E+00	2.104134420E+00
-2.789423297E+00	7.003032641E+00
5.128935670E+00	1.447635953E+00
1.135255271E+00	-4.850548045E+00
-1.066537767E+00	-3.060903499E+00
-4.827465934E+00	-1.223838564E-01
1.011479539E+00	4.876067761E+00
8.211875056E+00	3.120731360E+00
4.771525416E+00	-6.729981505E+00
-3.328005113E+00	1.278369529E+00
-2.613126577E+00	-2.592076424E-01
-1.819200016E+00	3.553610963E+00

TRUNCATED VALUES

-3.890000000E+00	2.100000000E+00
-2.780000000E+00	7.000000000E+00
5.120000000E+00	1.440000000E+00
1.130000000E+00	-4.850000000E+00
-1.060000000E+00	-3.060000000E+00
-4.820000000E+00	-1.200000000E-01
1.010000000E+00	4.870000000E+00
8.210000000E+00	3.120000000E+00
4.770000000E+00	-6.720000000E+00
-3.320000000E+00	1.270000000E+00
-2.610000000E+00	-2.500000000E-01
-1.810000000E+00	3.550000000E+00

TABLE 3B

COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 3.974336422E-08

COMPUTED CHARACTERISTIC VALUES

-1.749999976E-01	λ_k 1.200000005E+00	-8.500000306E-02	3.000000001E+00
-1.249999984E-01	3.999999996E+00	-1.000000044E-01	5.000000004E+00
-3.000000045E-01	9.000000001E+00	-2.000000000E-01	1.000000000E+01

CHARACTERISTIC VALUE DIFFERENCES

-2.352000000E-09	-5.100000000E-09	3.057100000E-09	-1.050000000E-09
-1.553000000E-09	4.140000000E-09	4.433000000E-09	-4.150000000E-09
4.516000000E-09	-6.800000000E-10	-4.200000000E-11	1.000000000E-10

COMPUTED COEFFICIENTS

-1.736481425E-01	b_k 9.848877481E-01	3.428201612E-01	9.396926483E-01
9.396926191E-01	-3.428201077E-01	-8.668254215E-01	-4.999999872E-01
-7.668444620E-01	-6.427876192E-01	-9.396926210E-01	3.428201421E-01

COEFFICIENT DIFFERENCES

-3.552500000E-08	4.922000000E-09	-1.822000000E-08	-2.725300000E-08
1.857000000E-09	-3.526400000E-08	1.753400000E-08	-1.282000000E-08
1.896200000E-08	9.227000000E-09	4.680000000E-11	9.170000000E-10

TABLE 3C

COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 3.989296129E-08

COMPUTED CHARACTERISTIC VALUES

-1.748585789E-01	λ_k 1.199989114E+00	-8.510682758E-02	3.888892925E+00
-1.251858584E-01	4.888179343E+00	-1.883142894E-01	4.999558749E+00
-2.983978899E-01	9.888834954E+00	-1.997986788E-01	1.899997588E+01

CHARACTERISTIC VALUE DIFFERENCES

-1.494291828E-04	9.888681888E-05	1.868274961E-04	-9.292498888E-05
1.858584248E-04	-1.793427288E-04	3.142893558E-04	4.492588988E-04
-1.682198872E-03	-8.349541988E-04	-2.893292418E-04	2.491788888E-05

COMPUTED COEFFICIENTS

-1.738778722E-01	b_k 9.838853988E-01	3.438151793E-01	9.393933358E-01
9.381688229E-01	-3.418689838E-01	-8.652325784E-01	-4.997842553E-01
-7.636856728E-01	-6.377888887E-01	-9.374728553E-01	3.421273877E-01

COEFFICIENT DIFFERENCES

2.296941728E-04	1.882362966E-03	-9.958362668E-04	2.992852438E-04
1.532598893E-03	-1.512399768E-04	-7.928256368E-04	-2.157444718E-04
-2.438778223E-03	-5.879681381E-03	-2.228565692E-03	-1.871646898E-04

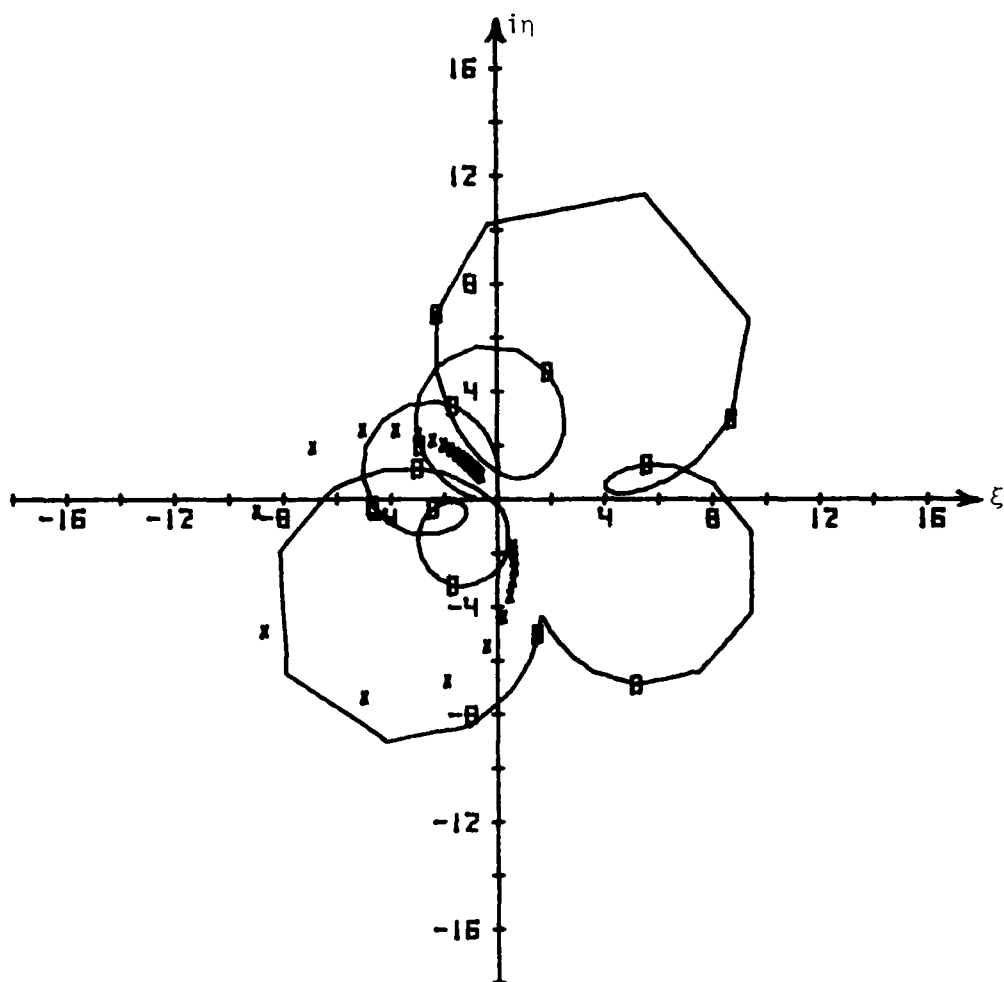


Figure 9a. Graph of $w(\omega)$, Coefficient $b_4 = -0.866025404 - 0.5i$

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DETERMINATION OF COMPLEX CHARACTERISTIC VALUES AND VECTORS FROM--ETC(U)
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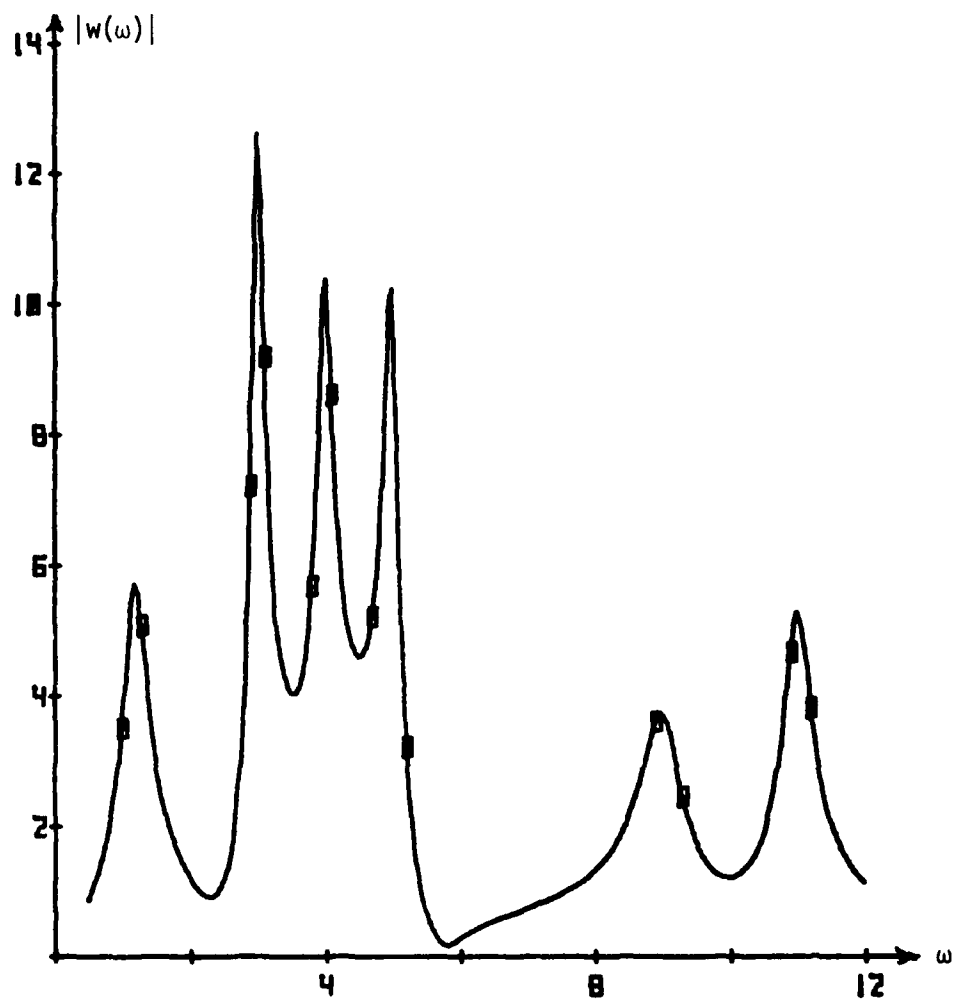


Figure 9b. Graph of $|w(\omega)|$, Coefficient $b_4 = -0.866025404 - 0.5i$

TABLE 4A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$b_4 = -0.866025404 + 0.5i$$

CHARACTERISTIC VALUES

 λ_k

-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	5.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.00000000E+01

COEFFICIENTS

 b_k

-1.73648170E-01	9.84807753E-01	3.42020143E-01	9.39692621E-01
9.39692621E-01	-3.42020143E-01	-8.66825404E-01	5.00000000E-01
-7.66044438E-01	-6.42787610E-01	-9.39692621E-01	3.42020143E-01

FREQUENCIES

 ω_p ω_{p+6}

1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
3.80000000E+00	4.10000000E+00
4.70000000E+00	5.20000000E+00
8.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01

FUNCTION VALUES

-4.30836699E+00	2.18760351E+00
-3.39189041E+00	7.02405502E+00
4.18772771E+00	1.51531031E+00
-1.96782655E+00	-3.85161154E+00
-1.68223470E+00	-3.05485074E+00
-4.72091367E+00	-1.19907482E-01
5.82716369E-01	4.88004813E+00
7.56357520E+00	3.14683174E+00
3.56400759E+00	-6.60923772E+00
5.73165093E-01	3.27740045E+00
-2.45062079E+00	-2.54291234E-01
-1.71967760E+00	3.55583071E+00

TRUNCATED VALUES

-4.30000000E+00	2.10000000E+00
-3.39000000E+00	7.02000000E+00
4.18000000E+00	1.51000000E+00
-1.96000000E+00	-3.85000000E+00
-1.68000000E+00	-3.05000000E+00
-4.72000000E+00	-1.10000000E-01
5.80000000E-01	4.88000000E+00
7.56000000E+00	3.14000000E+00
3.56000000E+00	-6.60000000E+00
5.70000000E-01	3.27000000E+00
-2.45000000E+00	-2.50000000E-01
-1.71000000E+00	3.55000000E+00

TABLE 4B

COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 4.384594668E-08

COMPUTED CHARACTERISTIC VALUES

-1.749999970E-01	1.200000009E+00	-8.500000293E-02	3.000000003E+00
-1.250000017E-01	3.999999995E+00	-1.000000071E-01	5.000000003E+00
-3.000000026E-01	9.000000004E+00	-2.000000009E-01	1.100000000E+01

CHARACTERISTIC VALUE DIFFERENCES

-2.991000000E-09	-9.150000000E-09	2.927200000E-09	-3.120000000E-09
1.715000000E-09	5.490000000E-09	7.114000000E-09	-2.820000000E-09
2.587000000E-09	-3.500000000E-09	9.220000000E-10	-2.000000000E-10

COMPUTED COEFFICIENTS

-1.736481104E-01	9.040077497E-01	3.420201821E-01	9.396926410E-01
9.396926470E-01	-3.420201061E-01	-8.668254100E-01	5.000000267E-01
-7.660444617E-01	-6.427876820E-01	-9.396926256E-01	3.420201470E-01

COEFFICIENT DIFFERENCES

-6.763000000E-08	3.299000000E-09	-3.913700000E-08	-2.001700000E-08
-2.683300000E-08	-3.686300000E-08	6.778000000E-09	-2.665000000E-08
1.871000000E-08	-8.013000000E-09	4.639000000E-09	-3.935000000E-09

TABLE 4C

COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 12

RADIUS OF DIFFERENCES 4.263717385E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.748875210E-01	1.288383221E+00	-8.496628959E-02	2.999974946E+00
-1.249392160E-01	4.000078669E+00	-9.976011942E-02	4.999918559E+00
-2.997245895E-01	9.000642855E+00	-1.999536272E-01	1.099948277E+01

CHARACTERISTIC VALUE DIFFERENCES

	-3.832214000E-04	-3.371041210E-05	2.585485000E-05
-1.124789950E-04	-7.066982000E-05	-2.398805790E-04	8.144870000E-05
-6.078403500E-05	-6.428553500E-04	-4.637282500E-05	5.172342000E-04
-2.754904950E-04			

COMPUTED COEFFICIENTS

b_k			
-1.727340771E-01	9.831958977E-01	3.418882556E-01	9.390710693E-01
9.374784685E-01	-3.414593559E-01	-8.648355125E-01	4.975483940E-01
-7.668064915E-01	-6.399448237E-01	-9.382810205E-01	3.416543817E-01

COEFFICIENT DIFFERENCES

	1.611855279E-03	1.318873810E-04	6.215516510E-04
-9.141008520E-04	-5.607871100E-04	-1.189891550E-03	2.451685900E-03
2.214152546E-03	-2.842786338E-03	-1.411600468E-03	3.657612920E-04
4.204852100E-05			

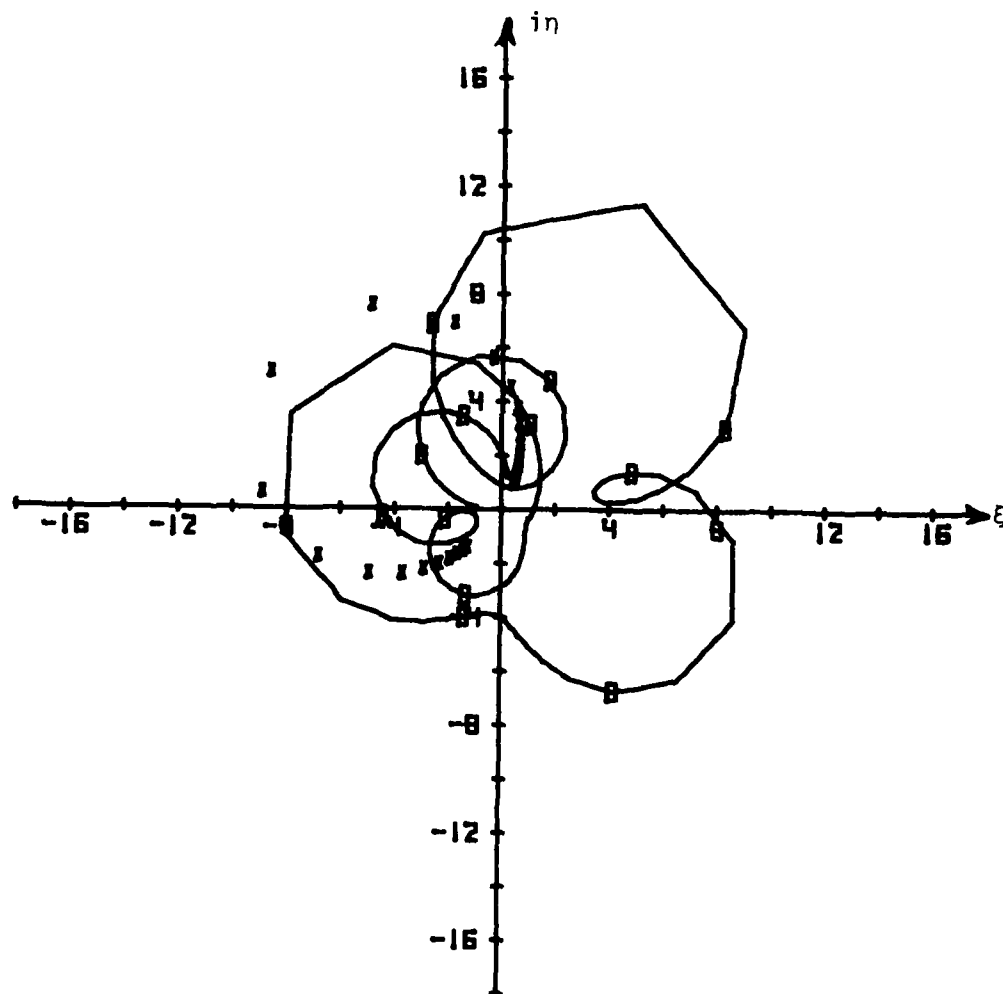


Figure 10a. Graph of $w(\omega)$, Coefficient $b_4 = -0.866025404 + 0.5i$

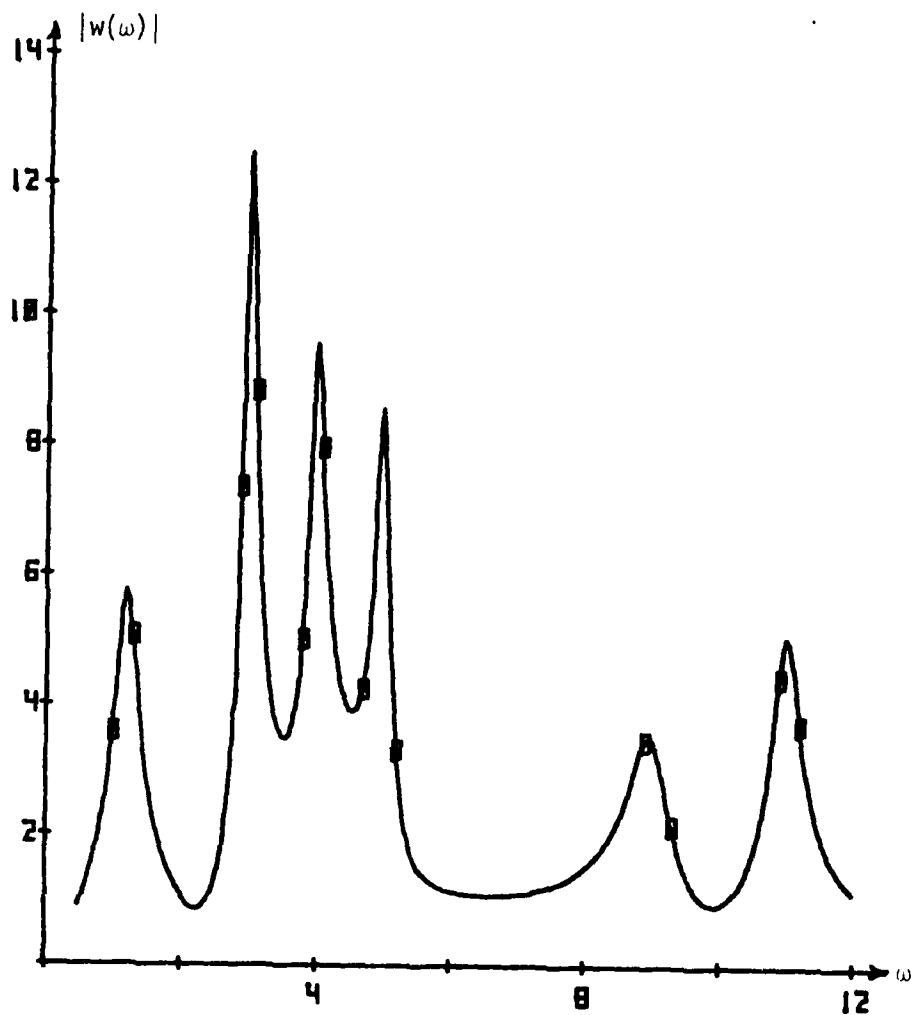


Figure 10b. Graph of $|w(\omega)|$, Coefficient $b_4 = -0.866025404 + 0.5i$

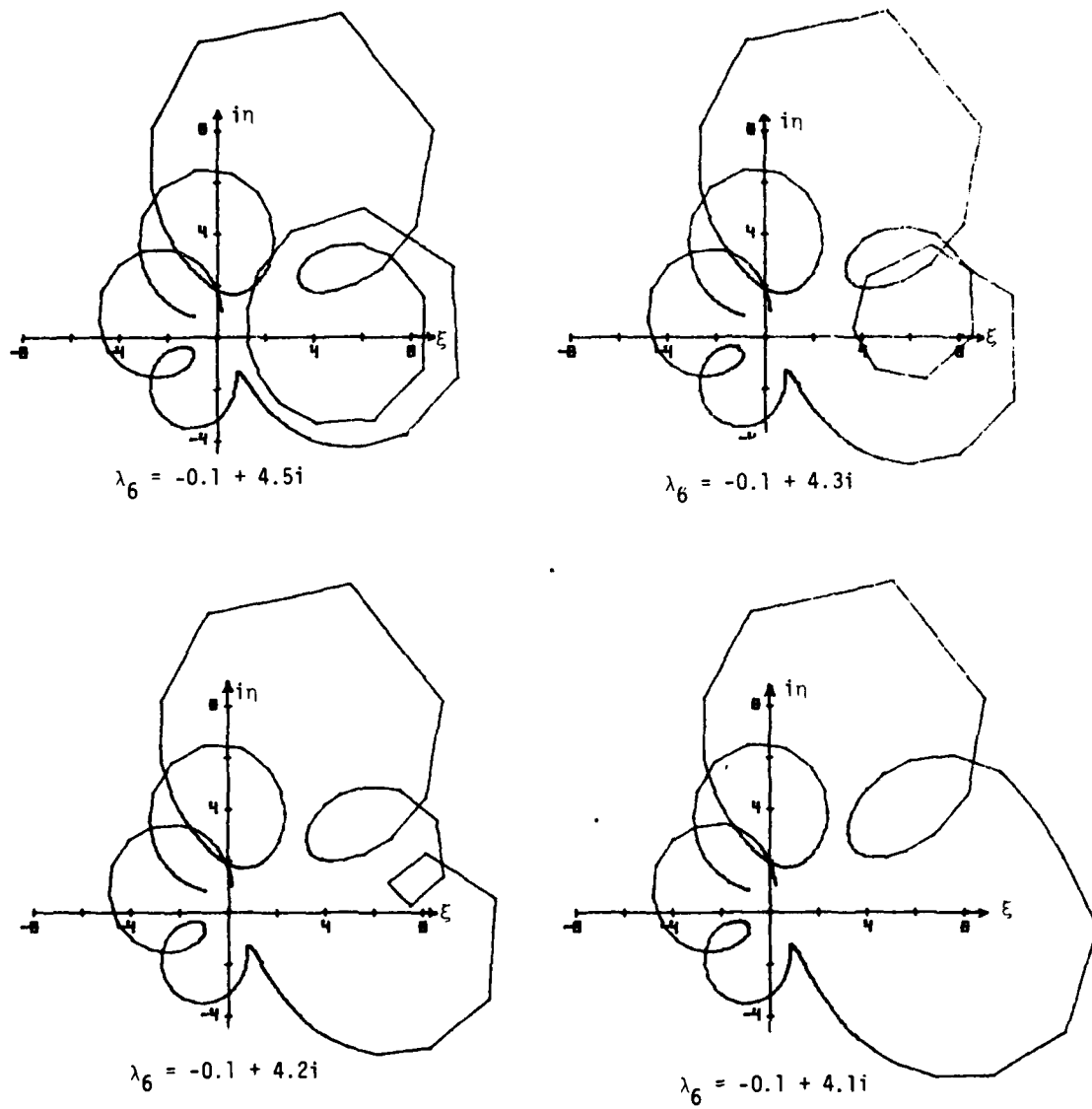


Figure 11a. $w(\omega)$ for Four Values of λ_6

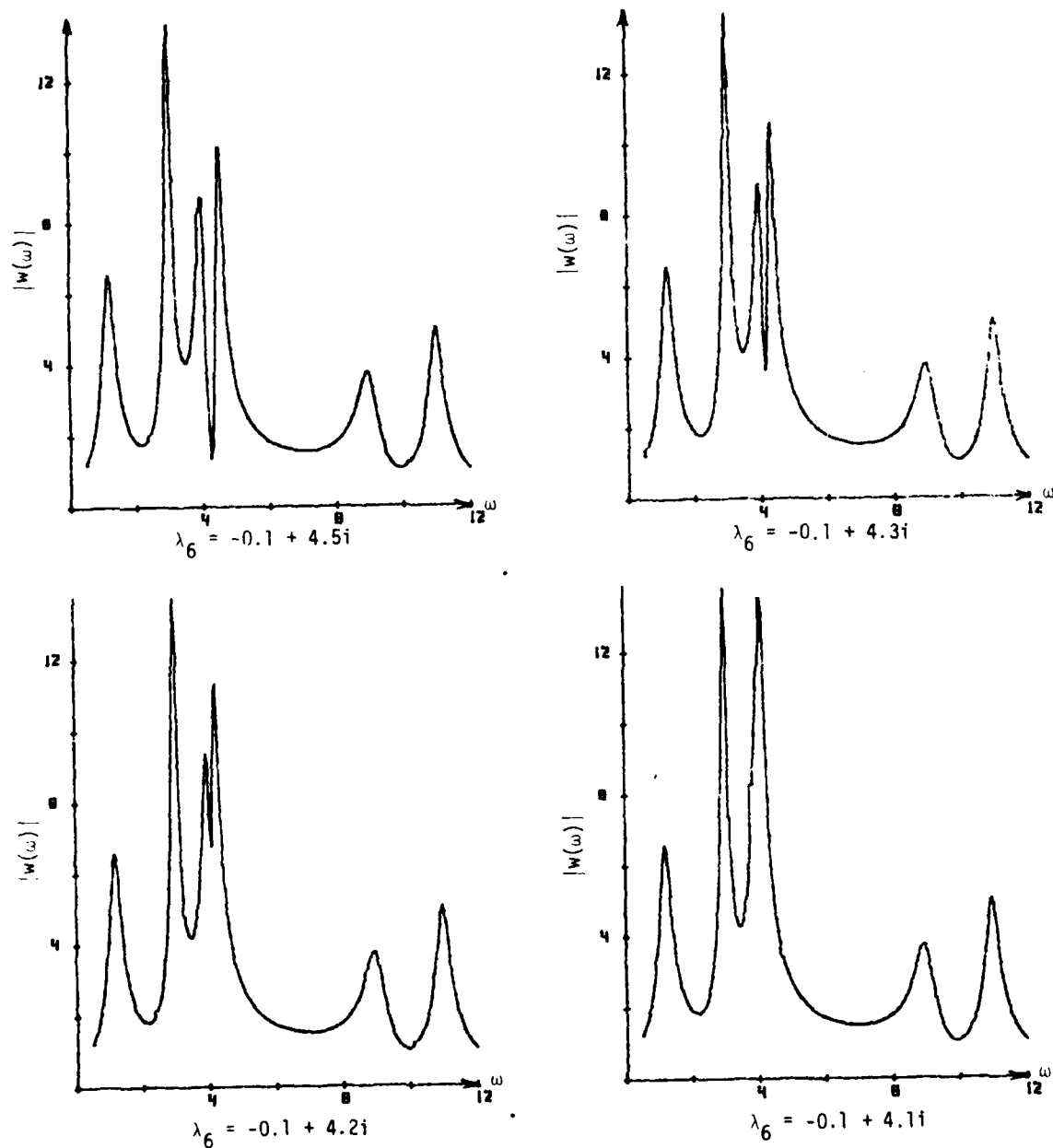


Figure 11b. Graphs of $|w(\omega)|$ for the Functions $w(\omega)$ of Figure 11a

TABLE 5A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$\lambda_6 = -0.1 + 4.5i$$

CHARACTERISTIC VALUES

 λ_k

-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	4.50000000E+00

COEFFICIENTS

 b_k

-1.73648178E-01	9.84807753E-01	3.42820143E-01	9.39692621E-01
-7.66844443E-01	-6.42787610E-01	-9.39692621E-01	3.42820143E-01
9.39692621E-01	-3.42820143E-01	8.66825484E-01	5.00000000E-01

FREQUENCIES

 ω_p ω_{p+6}

1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
8.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01
3.80000000E+00	4.10000000E+00
4.30000000E+00	4.70000000E+00

FUNCTION VALUES

TRUNCATED VALUES

-4.31681453E+00	2.27819543E+00	-4.31000000E+00	2.27000000E+00
-3.41284069E+00	7.75689897E+00	-3.41000000E+00	7.75000000E+00
-1.68782839E+00	-3.68113441E+00	-1.68000000E+00	-3.68000000E+00
-4.72324139E+00	-5.12859233E-01	-4.72000000E+00	-5.10000000E-01
4.13348557E+00	3.38716915E+00	4.13000000E+00	3.38000000E+00
6.96790764E-01	8.25234199E-01	6.96000000E-01	8.20000000E-01
5.74344548E-01	5.09940901E+00	5.70000000E-01	5.09000000E+00
7.53731997E+00	4.08745859E+00	7.53000000E+00	4.08000000E+00
-2.45466560E+00	-7.59789278E-01	-2.45000000E+00	-7.50000000E-01
-1.72181201E+00	3.17814272E+00	-1.72000000E+00	3.17000000E+00
3.55046825E+00	-3.58378854E+00	3.55000000E+00	-3.58000000E+00
4.12939254E+00	-4.48188982E+00	4.12000000E+00	-4.48000000E+00

INITIAL VALUE FOR UPPER CLOSE CHARACTERISTIC VALUE

$$X(6,1) = -2.00000000E-02 \quad X(6,2) = 4.70000000E+00$$

TABLE 5B
COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 14

RADIUS OF DIFFERENCES 3.290920758E-08

COMPUTED CHARACTERISTIC VALUES

-1.750000000E-01	λ_k 1.200000005E+00	-8.500000050E-02	3.000000002E+00
-2.999999980E-01	9.000000004E+00	-2.000000011E-01	1.100000000E+01
-1.250000006E-01	3.999999999E+00	-9.999999976E-02	4.499999998E+00

CHARACTERISTIC VALUE DIFFERENCES

8.030000000E-10	-5.260000000E-09	4.956000000E-10	-1.630000000E-09
-1.973000000E-09	-4.240000000E-09	1.000000000E-09	2.000000000E-10
6.300000000E-10	1.160000000E-09	-2.370000000E-10	2.170000000E-09

COMPUTED COEFFICIENTS

-1.736481404E-01	b_k 9.848077649E-01	3.420201611E-01	9.396926210E-01
-7.660444471E-01	-6.427875873E-01	-9.396926283E-01	3.420201452E-01
9.396926321E-01	-3.420201350E-01	8.660253906E-01	5.000000113E-01

COEFFICIENT DIFFERENCES

-3.755600000E-08	-1.189800000E-08	-1.809500000E-08	-1.500000000E-11
4.125000000E-09	-2.267400000E-08	7.275000000E-09	-2.180000000E-09
-1.109600000E-08	-8.012000000E-09	5.400000000E-09	-1.132000000E-08

TABLE 5C
COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 14

RADIUS OF DIFFERENCES 3.319800208E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.752347025E-01	1.199923297E+00	-8.496740030E-02	2.999972497E+00
-2.982009368E-01	9.000870285E+00	-2.002315172E-01	1.099951521E+01
-1.252153949E-01	4.000004482E+00	-9.969653607E-02	4.499991346E+00

CHARACTERISTIC VALUE DIFFERENCES

2.247025070E-04	7.670301000E-05	-3.259969820E-05	2.750266000E-05
-1.799063161E-03	-8.702845800E-04	2.315171860E-04	4.847946000E-04
2.153948530E-04	-8.448213000E-05	-3.034639290E-04	8.654040000E-06

COMPUTED COEFFICIENTS

b_k			
-1.735835646E-01	9.839545396E-01	3.417477000E-01	9.385999003E-01
-7.632716026E-01	-6.369041343E-01	-9.401204511E-01	3.402979259E-01
9.395063947E-01	-3.421755437E-01	8.641210258E-01	4.992500052E-01

COEFFICIENT DIFFERENCES

-6.461337600E-05	8.532133500E-04	2.724430200E-04	1.092720725E-03
-2.772840382E-03	-5.083475717E-03	4.278300660E-04	1.722217148E-03
1.862262730E-04	1.554007400E-04	1.904378190E-03	7.499948500E-04

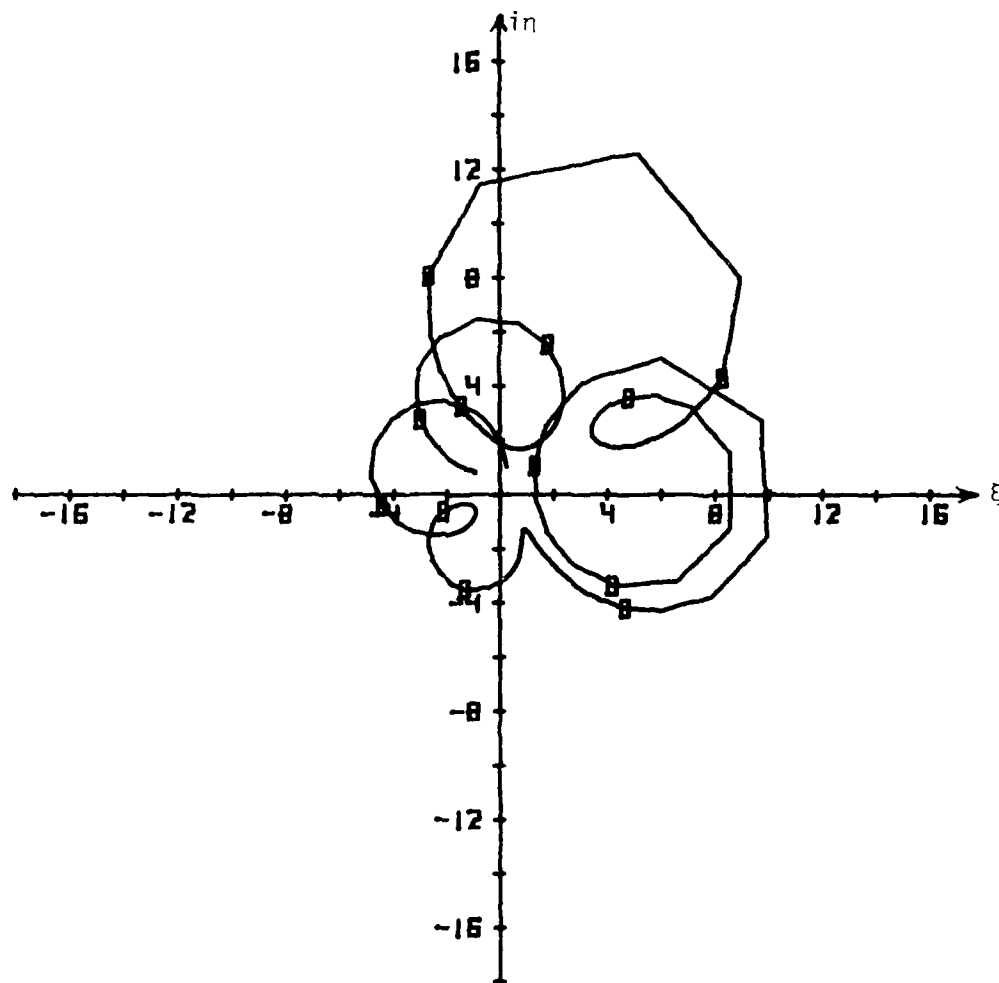


Figure 12a. Graph of $w(\omega)$, $\lambda_6 = -0.1 + 4.5i$

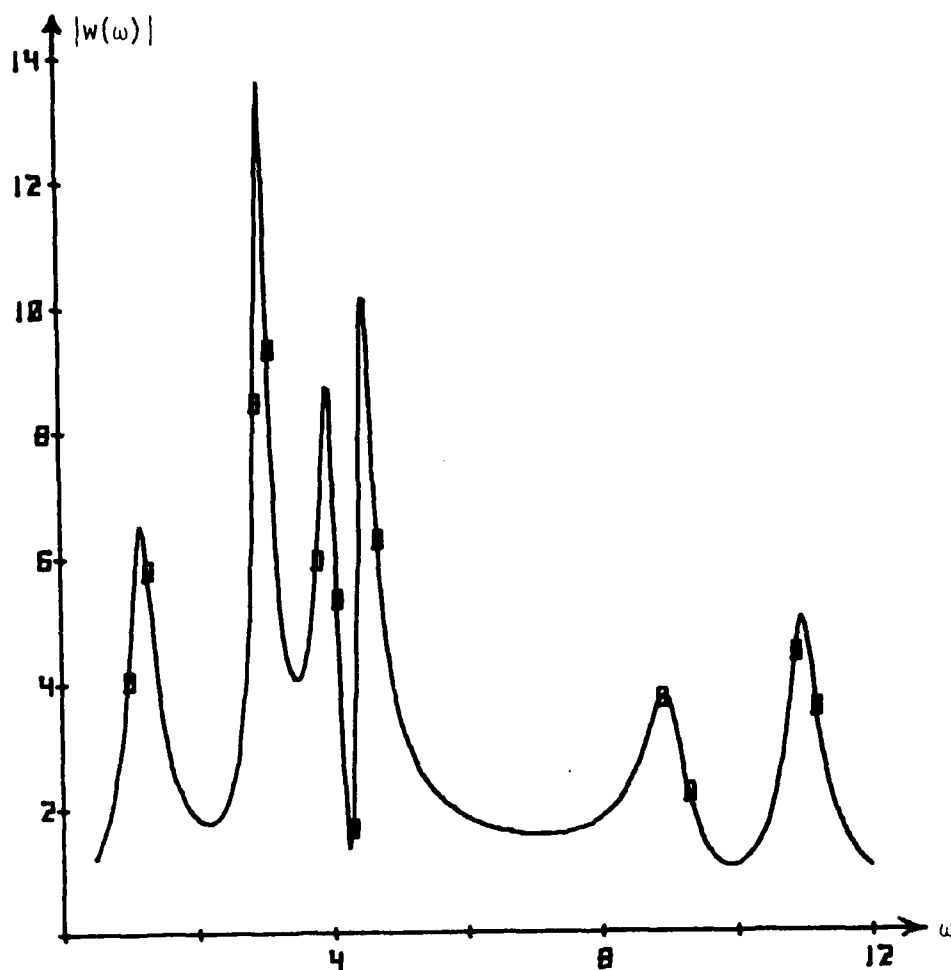


Figure 12b. Graph of $|w(\omega)|$
 $\lambda_6 = -0.1 + 4.5i$

Figure 12b. Graph of $|w(\omega)|$, $\lambda_6 = -0.1 + 4.5i$

TABLE 6A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$\lambda_6 = -0.1 + 4.3i$$

CHARACTERISTIC VALUES

 λ_k

-1. 75000000E-01	1. 20000000E+00	-8. 50000000E-02	3. 00000000E+00
-3. 00000000E-01	9. 00000000E+00	-2. 00000000E-01	1. 10000000E+01
-1. 25000000E-01	4. 00000000E+00	-1. 00000000E-01	4. 30000000E+00

COEFFICIENTS

 b_k

-1. 73648178E-01	9. 84887753E-01	3. 42020143E-01	9. 39692621E-01
-7. 66044443E-01	-6. 42787510E-01	-9. 39692621E-01	3. 42020143E-01
9. 39692621E-01	-3. 42020143E-01	8. 66025404E-01	5. 00000000E-01

FREQUENCIES ω_p ω_{p+6}

1. 00000000E+00	1. 30000000E+00
2. 90000000E+00	3. 10000000E+00
8. 90000000E+00	9. 30000000E+00
1. 09000000E+01	1. 12000000E+01
3. 80000000E+00	4. 10000000E+00
4. 20000000E+00	4. 50000000E+00

FUNCTION VALUES

TRUNCATED VALUES

-4. 32697570E+00	2. 27959870E+00	-4. 32000000E+00	2. 27000000E+00
-3. 44841017E+00	7. 83500576E+00	-3. 44000000E+00	7. 83000000E+00
-1. 69289246E+00	-3. 59379657E+00	-1. 69000000E+00	-3. 59000000E+00
-4. 72615075E+00	-5. 09579901E-01	-4. 72000000E+00	-5. 00000000E-01
4. 03032478E+00	3. 84986321E+00	4. 03000000E+00	3. 84000000E+00
3. 60096605E+00	2. 10064881E+00	3. 60000000E+00	2. 10000000E+00
5. 62234297E-01	5. 11263045E+00	5. 60000000E-01	5. 11000000E+00
7. 49292700E+00	4. 11469628E+00	7. 49000000E+00	4. 11000000E+00
-2. 45964077E+00	-7. 53681921E-01	-2. 45000000E+00	-7. 50000000E-01
-1. 72448563E+00	3. 18110635E+00	-1. 72000000E+00	3. 18000000E+00
3. 94822765E+00	-1. 45394061E+00	3. 94000000E+00	-1. 45000000E+00
4. 24784072E+00	-4. 95205959E+00	4. 24000000E+00	-4. 95000000E+00

INITIAL VALUE FOR UPPER CLOSE CHARACTERISTIC VALUE

$$X(6.1) = -2. 00000000E-02 \quad X(6.2) = 4. 50000000E+00$$

TABLE 6B

COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 15

RADIUS OF DIFFERENCES 2.028380666E-08

COMPUTED CHARACTERISTIC VALUES

-1.750000002E-01	λ_k 1.200000003E+00	-8.500000035E-02	3.000000001E+00
-3.000000000E-01	9.000000003E+00	-2.000000007E-01	1.100000000E+01
-1.250000019E-01	3.999999999E+00	-9.99999856E-02	4.299999999E+00

CHARACTERISTIC VALUE DIFFERENCES

1.000000000E-10	-3.000000000E-09	3.506000000E-10	-1.490000000E-09
5.000000000E-12	-2.770000000E-09	6.700000000E-10	3.000000000E-10
1.917000000E-09	7.000000000E-10	-1.443000000E-09	1.150000000E-09

COMPUTED COEFFICIENTS

-1.736481548E-01	b_k 9.848077581E-01	3.420201592E-01	9.396926202E-01
-7.660444500E-01	-6.427875986E-01	-9.396926259E-01	3.420201433E-01
9.396926410E-01	-3.420201325E-01	8.660253875E-01	5.000000064E-01

COEFFICIENT DIFFERENCES

-2.321300000E-08	-5.053000000E-09	-1.615400000E-08	8.370000000E-10
7.845000000E-09	-1.144000000E-08	4.863000000E-09	-2.720000000E-10
-2.001200000E-08	-1.052500000E-08	1.652000000E-08	-6.370000000E-09

TABLE 6C
COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 15

RADIUS OF DIFFERENCES 2.002837512E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.747504065E-01	1.200245470E+00	-8.505172644E-02	3.000031437E+00
-2.987863957E-01	8.999429618E+00	-2.003951963E-01	1.100012132E+01
-1.250158041E-01	3.999854478E+00	-1.000991056E-01	4.300193887E+00

CHARACTERISTIC VALUE DIFFERENCES

-2.495934730E-04	-2.454696800E-04	5.172643570E-05	-3.143725000E-05
-1.213604269E-03	5.701822600E-04	3.951963360E-04	-1.213162000E-04
1.580405300E-05	1.455221200E-04	9.910564000E-05	-1.938869900E-04

COMPUTED COEFFICIENTS

b_k			
-1.733584277E-01	9.824028985E-01	3.424241207E-01	9.390667146E-01
-7.607610209E-01	-6.399357628E-01	-9.394357272E-01	3.442706322E-01
9.371556185E-01	-3.418099093E-01	8.661640285E-01	4.975241475E-01

COEFFICIENT DIFFERENCES

-2.897502600E-04	2.404854469E-03	-4.039776930E-04	6.259063920E-04
-5.283422139E-03	-2.851847210E-03	-2.568938260E-04	-2.250489156E-03
2.537002473E-03	-2.102336730E-04	-1.386245400E-04	2.475852470E-03

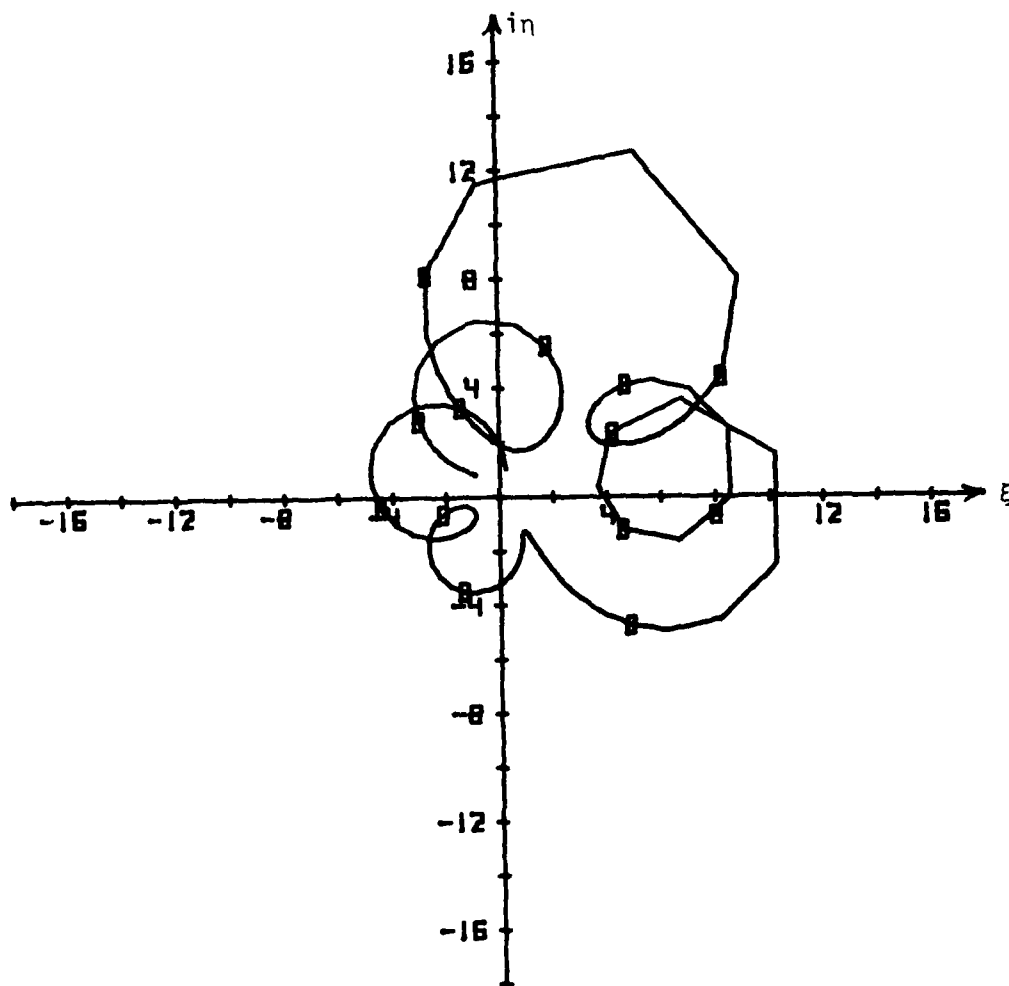


Figure 13a. Graph of $w(\omega)$, $\lambda_6 = -0.1 + 4.3i$

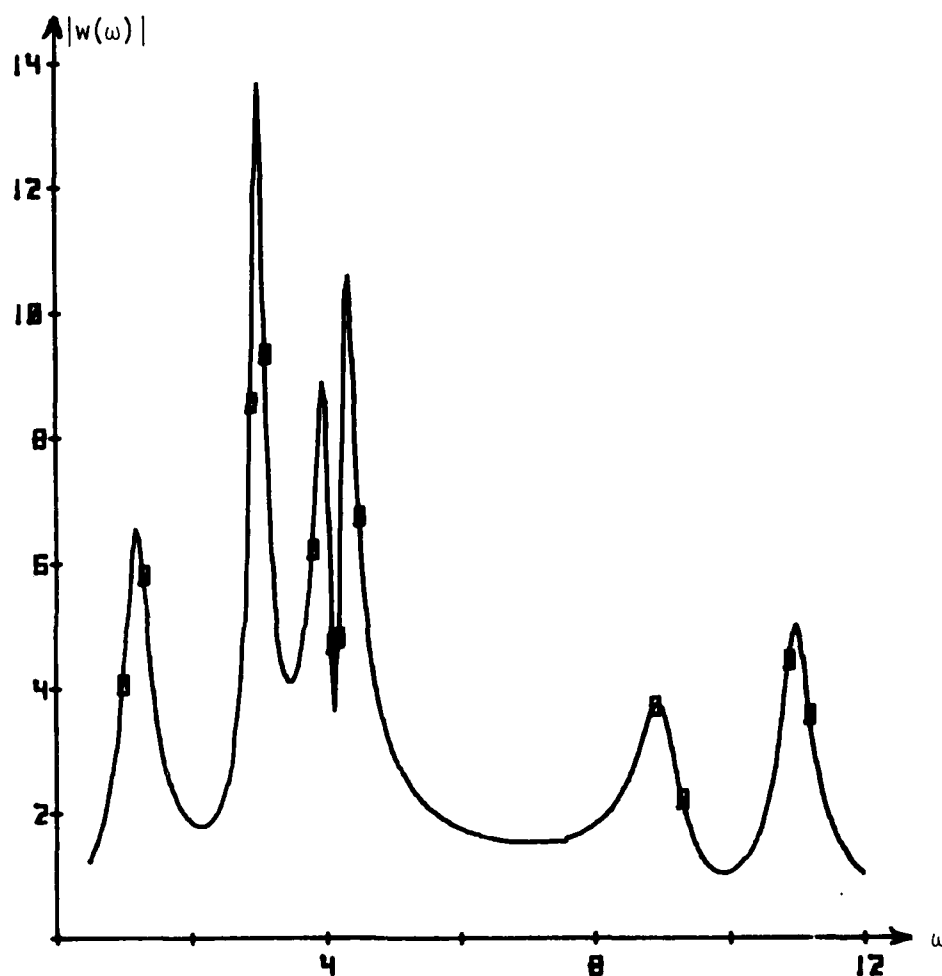


Figure 13b. Graph of $|w(\omega)|$ $\lambda_6 = -0.1 + 4.3i$

TABLE 7A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$\lambda_6 = -0.1 + 4.2i$$

CHARACTERISTIC VALUES

λ_k			
-1. 75000000E-01	1. 20000000E+00	-8. 50000000E-02	3. 00000000E+00
-3. 00000000E-01	9. 00000000E+00	-2. 00000000E-01	1. 10000000E+01
-1. 25000000E-01	4. 00000000E+00	-1. 00000000E-01	4. 20000000E+00

COEFFICIENTS

b_k			
-1. 73648178E-01	9. 84807753E-01	3. 42020143E-01	9. 39692621E-01
-7. 66044443E-01	-6. 42787610E-01	-9. 39692621E-01	3. 42020143E-01
9. 39692621E-01	-3. 42020143E-01	8. 66025404E-01	5. 00000000E-01

FREQUENCIES ω_p

ω_p	ω_{p+6}
1. 00000000E+00	1. 30000000E+00
2. 90000000E+00	3. 10000000E+00
8. 90000000E+00	9. 30000000E+00
1. 09000000E+01	1. 12000000E+01
3. 80000000E+00	3. 90000000E+00
4. 10000000E+00	4. 40000000E+00

FUNCTION VALUES

-4. 33288590E+00	2. 28485934E+00
-3. 46938132E+00	7. 88412155E+00
-1. 69565556E+00	-3. 59040033E+00
-4. 72755240E+00	-5. 00036922E-01
3. 99099462E+00	4. 32259153E+00
6. 04561676E+00	9. 10026001E-01
5. 55659942E-01	5. 12007134E+00
7. 46627877E+00	4. 18370156E+00
-2. 46199895E+00	-7. 50842972E-01
-1. 72577650E+00	3. 18250295E+00
6. 45884961E+00	4. 49098800E+00
4. 38752293E+00	-5. 41110482E+00

TRUNCATED VALUES

-4. 33000000E+00	2. 28000000E+00
-3. 46000000E+00	7. 88000000E+00
-1. 69000000E+00	-3. 59000000E+00
-4. 72000000E+00	-5. 00000000E-01
3. 99000000E+00	4. 32000000E+00
6. 04000000E+00	9. 10000000E-01
5. 50000000E-01	5. 12000000E+00
7. 46000000E+00	4. 18000000E+00
-2. 46000000E+00	-7. 50000000E-01
-1. 72000000E+00	3. 18000000E+00
6. 45000000E+00	4. 49000000E+00
4. 38000000E+00	-5. 41000000E+00

INITIAL VALUE FOR UPPER CLOSE CHARACTERISTIC VALUE

$$X(6,1) = -2. 00000000E-02 \quad X(6,2) = 4. 30000000E+00$$

TABLE 7B

COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 14

RADIUS OF DIFFERENCES 4.418275813E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.750000022E-01	1.200000000E+00	-8.500000054E-02	3.000000002E+00
-3.000000000E-01	9.000000005E+00	-2.000000013E-01	1.100000000E+01
-1.250000017E-01	4.000000001E+00	-9.999999732E-02	4.199999997E+00

CHARACTERISTIC VALUE DIFFERENCES

2.233000000E-09	-8.350000000E-09	5.425000000E-10	-2.290000000E-09
2.900000000E-11	-5.020000000E-09	1.287000000E-09	3.000000000E-10
1.663000000E-09	-5.400000000E-10	-2.680000000E-09	3.470000000E-09

COMPUTED COEFFICIENTS

b_k			
-1.736481202E-01	9.848077799E-01	3.420201680E-01	9.396926199E-01
-7.668444567E-01	-6.427875891E-01	-9.396926295E-01	3.420201451E-01
9.396926637E-01	-3.420201457E-01	8.668253690E-01	5.000000281E-01

COEFFICIENT DIFFERENCES

-5.778200000E-08	-2.686900000E-08	-2.495200000E-08	1.000000000E-09
1.369900000E-08	-2.088300000E-08	8.544000000E-09	-2.114000000E-09
-4.271000000E-08	2.676000000E-09	3.499000000E-08	-2.813000000E-08

TABLE 7C
COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 14

RADIUS OF DIFFERENCES 4.425468746E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.749991751E-01	1.200131623E+00	-8.583565927E-02	3.000026438E+00
-2.993761382E-01	9.000250163E+00	-2.002009081E-01	1.100010790E+01
-1.253803169E-01	4.000738998E+00	-9.938138200E-02	4.199427841E+00

CHARACTERISTIC VALUE DIFFERENCES

	-1.316227400E-04	3.565927050E-05	-2.643022000E-05
-8.249680000E-07	-2.501627800E-04	2.009081120E-04	-1.077952000E-04
-6.238617530E-04	-7.389976200E-04	-6.186179950E-04	5.721590300E-04
3.803168970E-04			

COMPUTED COEFFICIENTS

b_k			
-1.740718620E-01	9.838335283E-01	3.423628101E-01	9.385671182E-01
-7.637119049E-01	-6.413197966E-01	-9.383571265E-01	3.435831738E-01
9.465311470E-01	-3.471542339E-01	8.601462645E-01	5.055263387E-01

COEFFICIENT DIFFERENCES

	9.742247200E-04	-3.426671420E-04	1.125502835E-03
4.236840470E-04	-1.467813378E-03	-1.335494526E-03	-1.563038795E-03
-2.332538088E-03	5.134090889E-03	5.879139490E-03	-5.526338720E-03
-6.838525966E-03			

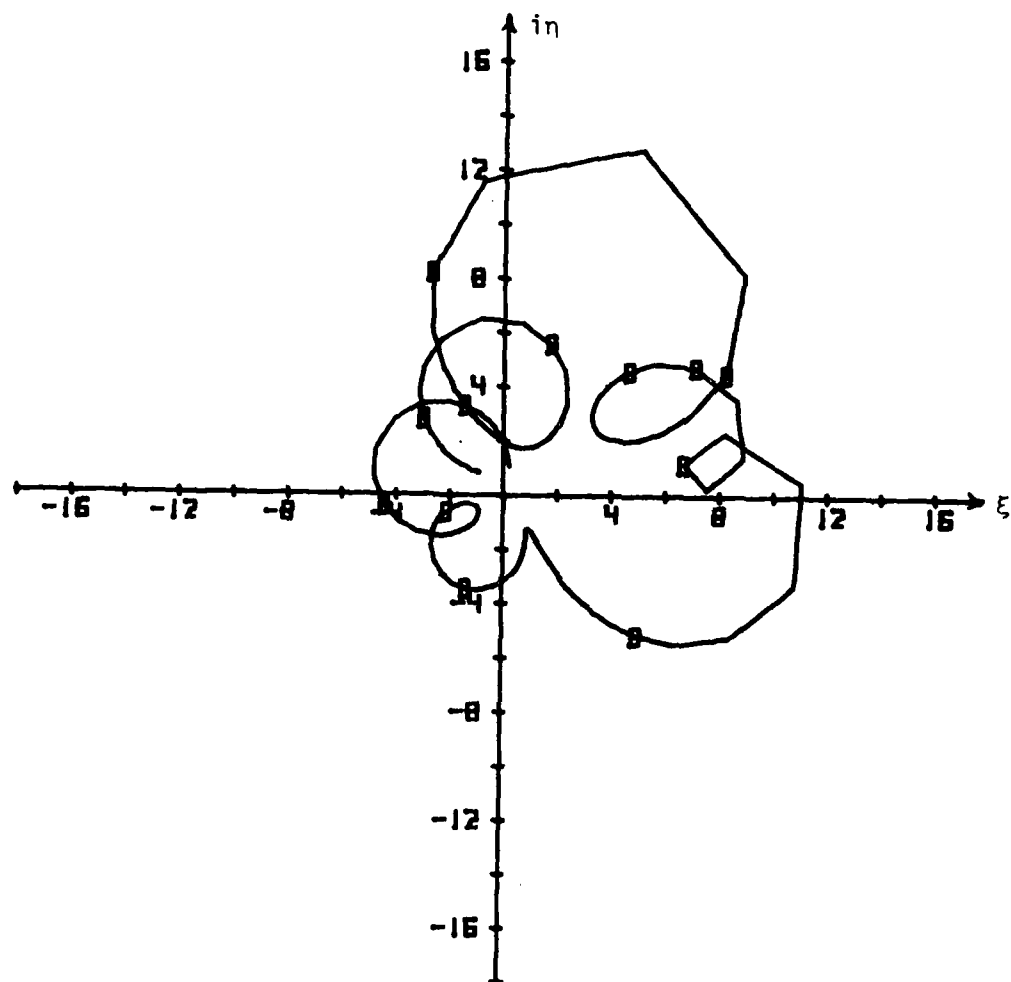


Figure 14a. Graph of $w(\omega)$, $\lambda_6 = -0.1 + 4.2i$

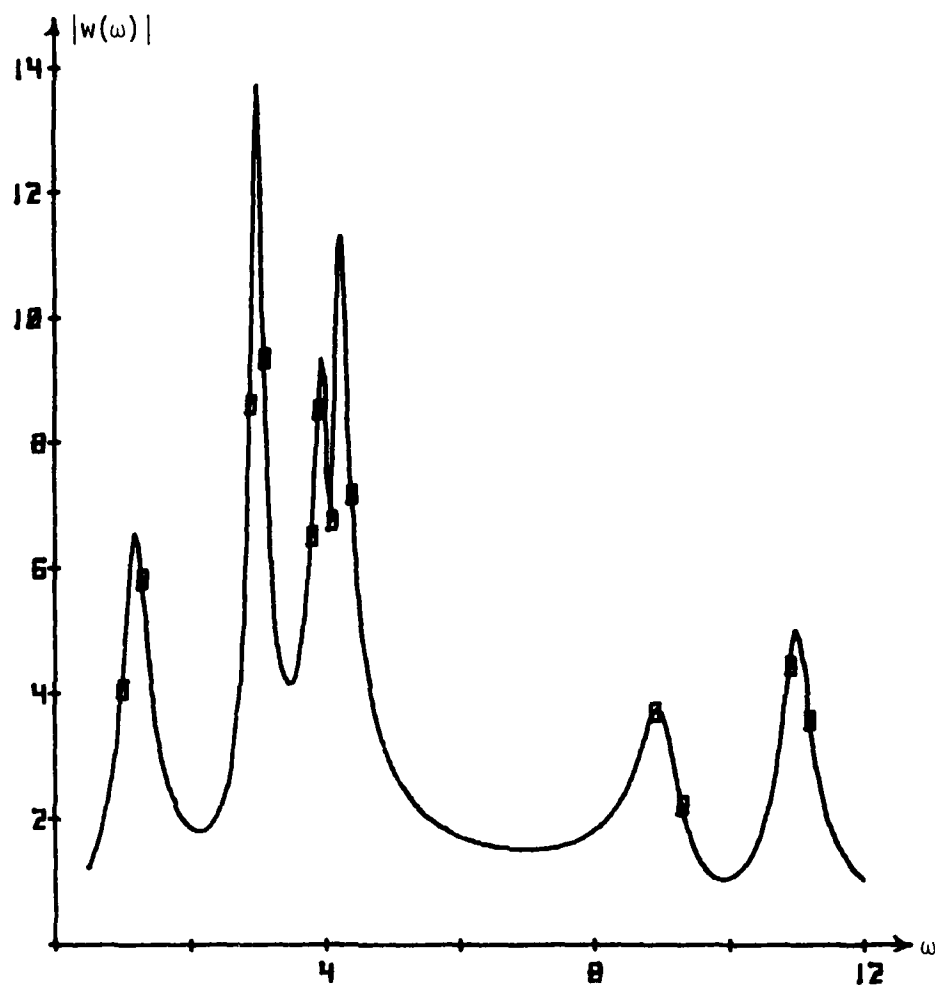


Figure 14b. Graph of $|w(\omega)|$, $\lambda_6 = -0.1 + 4.2i$

TABLE 8A
DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

$$\lambda_6 = 0.1 + 4.1i$$

CHARACTERISTIC VALUES

 λ_k

-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-2.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	4.10000000E+00

COEFFICIENTS

 b_k

-1.73648178E-01	9.84807753E-01	3.42020143E-01	9.39692621E-01
-7.66044443E-01	-6.42787610E-01	-9.39692621E-01	3.42020143E-01
9.39692621E-01	-3.42020143E-01	8.66025404E-01	5.00000000E-01

FREQUENCIES ω_p ω_{p+6}

1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
8.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01
3.80000000E+00	3.90000000E+00
4.20000000E+00	4.40000000E+00

FUNCTION VALUES

TRUNCATED VALUES

-4.33911418E+00	2.29054749E+00	-4.33000000E+00	2.29000000E+00
-3.49299427E+00	7.94187852E+00	-3.49000000E+00	7.94000000E+00
-1.69831619E+00	-3.58717350E+00	-1.69000000E+00	-3.58000000E+00
-4.72892107E+00	-5.06555065E-01	-4.72000000E+00	-5.00000000E-01
4.02330768E+00	5.08745382E+00	4.02000000E+00	5.08000000E+00
8.59960767E+00	-6.55499299E+00	8.59000000E+00	-6.55000000E+00
5.48696993E-01	5.12815432E+00	5.48000000E-01	5.12000000E+00
7.43590302E+00	4.26715826E+00	7.43000000E+00	4.26000000E+00
-2.46427818E+00	-7.48137040E-01	-2.46000000E+00	-7.40000000E-01
-1.72703874E+00	3.18334565E+00	-1.72000000E+00	3.18000000E+00
6.92413696E+00	5.85563840E+00	6.82000000E+00	5.85000000E+00
3.02084151E+00	-5.04627964E+00	3.02000000E+00	-5.04000000E+00

INITIAL VALUE FOR UPPER CLOSE CHARACTERISTIC VALUE

$$X(6,1) = -2.00000000E-02 \quad X(6,2) = 4.30000000E+00$$

TABLE 8B

COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 15

RADIUS OF DIFFERENCES 2.856670392E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.750000011E-01	1.200000006E+00	-8.500000122E-02	3.000000001E+00
-3.000000028E-01	9.000000000E+00	-1.999999999E-01	1.100000000E+01
-1.250000154E-01	3.999999990E+00	-1.000000059E-01	4.100000012E+00

CHARACTERISTIC VALUE DIFFERENCES

1.114000000E-09	-5.610000000E-09	1.224400000E-09	-1.230000000E-09
2.791000000E-09	2.000000000E-11	-1.450000000E-10	5.000000000E-10
1.542600000E-08	1.036000000E-08	5.911000000E-09	-1.181000000E-08

COMPUTED COEFFICIENTS

b_k			
-1.736481394E-01	9.848077685E-01	3.420201591E-01	9.396926300E-01
-7.660444544E-01	-6.427876174E-01	-9.396926213E-01	3.420201397E-01
9.396926154E-01	-3.420197904E-01	8.660253991E-01	4.999996645E-01

COEFFICIENT DIFFERENCES

-3.855200000E-08	-1.551700000E-08	-1.613000000E-08	-9.038000000E-09
1.141600000E-08	7.383000000E-09	3.730000000E-10	3.324000000E-09
5.603000000E-09	-3.525930000E-07	4.910000000E-09	3.355370000E-07

TABLE 8C
COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 15

RADIUS OF DIFFERENCES 3.020321145E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.752805004E-01	1.200160501E+00	-8.498616739E-02	2.999957827E+00
-2.988125622E-01	9.001144931E+00	-2.001353880E-01	1.099998211E+01
-1.239124132E-01	4.002052306E+00	-9.828075794E-02	4.098888774E+00

CHARACTERISTIC VALUE DIFFERENCES

3.805003680E-04	-1.605003680E-04	-1.383260930E-05	4.217287000E-05
-1.186437795E-03	-1.144931150E-03	1.353880440E-04	1.789110000E-05
-1.086586848E-03	-2.052305900E-03	-1.719242059E-03	1.191225970E-03

COMPUTED COEFFICIENTS

b_k			
-1.72920544E-01	9.841144826E-01	3.420238963E-01	9.389234399E-01
-7.637874777E-01	-6.365813508E-01	-9.383805521E-01	3.426750060E-01
9.649118792E-01	-3.832885728E-01	8.404534054E-01	5.413090141E-01

COEFFICIENT DIFFERENCES

-7.272235560E-04	6.932704270E-04	-3.753322000E-06	7.691810710E-04
-2.256965311E-03	-6.206259201E-03	-1.312068886E-03	-6.549430280E-04
-2.521925817E-02	4.126842982E-02	2.557199863E-02	-4.130901405E-02

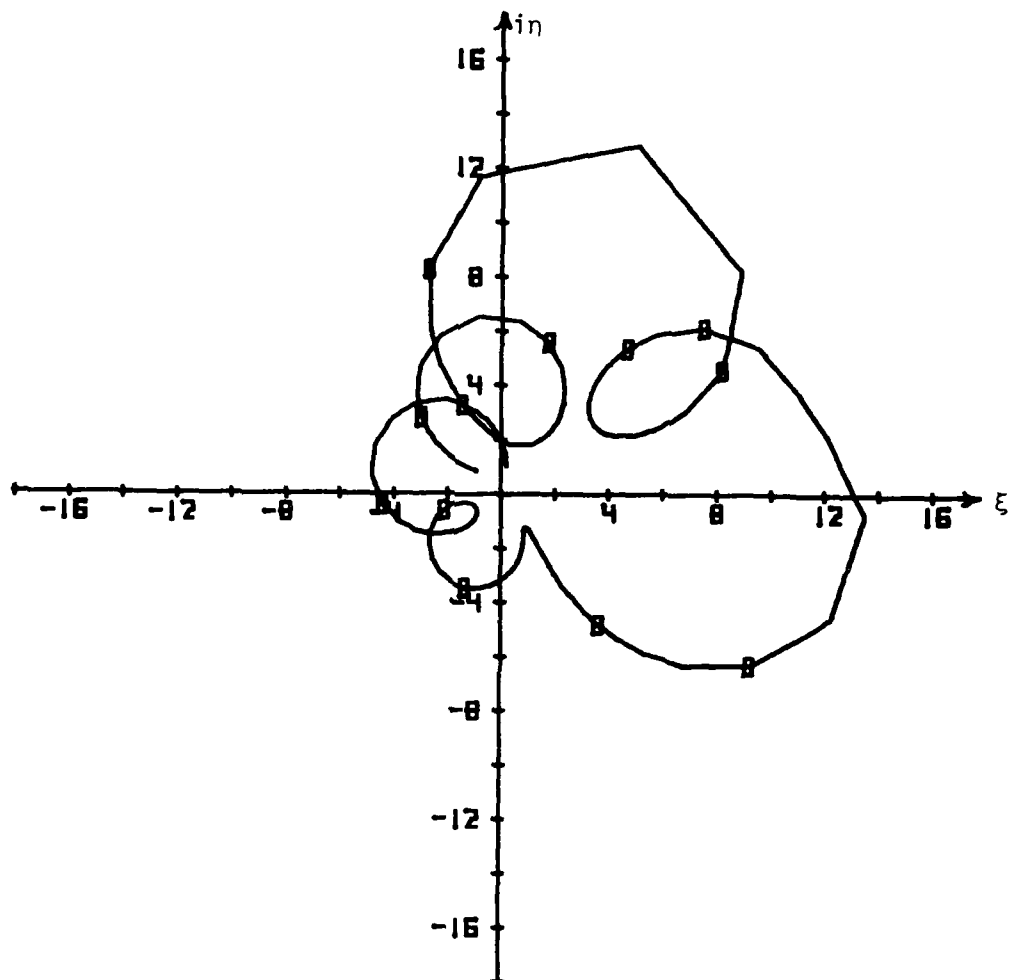


Figure 15a. Graph of $w(\omega)$, $\lambda_6 = -0.1 + 4.1i$

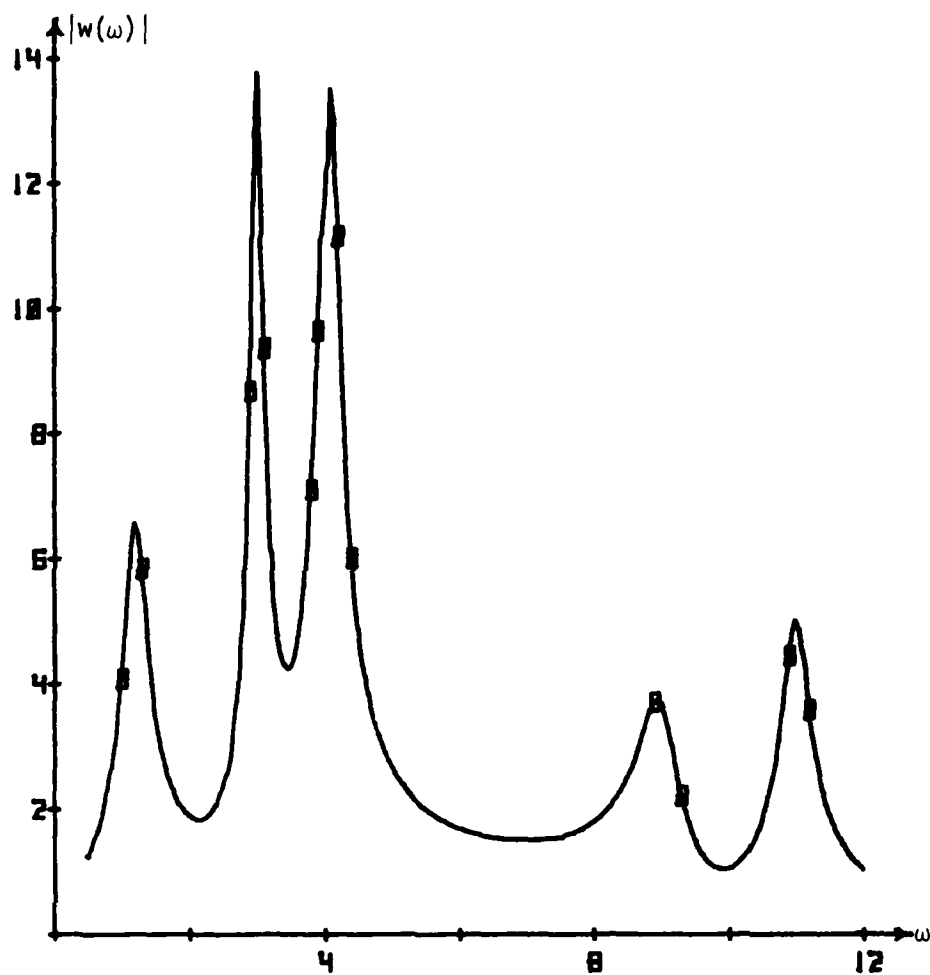


Figure 15b. Graph of $|w(\omega)|$, $\lambda_6 = -0.1 + 4.1i$

TABLE 9A

DEFINING PARAMETERS, FUNCTION VALUES AND TRUNCATED VALUES

CHARACTERISTIC VALUES

λ_k			
-1.75000000E-01	1.20000000E+00	-8.50000000E-02	3.00000000E+00
-3.00000000E-01	9.00000000E+00	-2.00000000E-01	1.10000000E+01
-1.25000000E-01	4.00000000E+00	-1.00000000E-01	4.05000000E+00

COEFFICIENTS

b_k			
-1.73648178E-01	9.84887753E-01	3.42828143E-01	9.39692621E-01
-7.66844443E-01	-6.42787610E-01	-9.39692621E-01	3.42828143E-01
9.39692621E-01	-3.42828143E-01	8.66825484E-01	5.00000000E-01

FREQUENCIES ω_p

ω_p	ω_{p+6}
1.00000000E+00	1.30000000E+00
2.90000000E+00	3.10000000E+00
9.90000000E+00	9.30000000E+00
1.09000000E+01	1.12000000E+01
3.80000000E+00	3.90000000E+00
4.20000000E+00	4.40000000E+00

FUNCTION VALUES

-4.34235688E+00	2.29356679E+00
-7.50594228E+00	7.97471804E+00
-1.69961824E+00	-3.58561980E+00
-4.72959356E+00	-5.05836331E-01
4.12727686E+00	5.66461914E+00
6.74151685E+00	-7.18318567E+00
5.45858426E-01	5.13245995E+00
7.41918448E+00	4.31575826E+00
-2.46538971E+00	-7.46831786E-01
-1.72765955E+00	3.18449734E+00
7.44871157E+00	6.92636828E+00
2.62884120E+00	-4.85987044E+00

TRUNCATED VALUES

-4.34000000E+00	2.29000000E+00
-7.50000000E+00	7.97000000E+00
-1.69000000E+00	-3.58000000E+00
-4.72000000E+00	-5.00000000E-01
4.12000000E+00	5.66000000E+00
6.74000000E+00	-7.18000000E+00
5.40000000E-01	5.13000000E+00
7.41000000E+00	4.31000000E+00
-2.46000000E+00	-7.40000000E-01
-1.72000000E+00	3.18000000E+00
7.44000000E+00	6.92000000E+00
2.62000000E+00	-4.85000000E+00

INITIAL VALUE FOR UPPER CLOSE CHARACTERISTIC VALUE

X(6,1) = -2.00000000E-02 X(6,2) = 4.20000000E+00

TABLE 9B
COMPUTED DEFINING PARAMETERS FROM "EXACT" FUNCTION VALUES

NUMBER OF ITERATIONS= 18

RADIUS OF DIFFERENCES 3.118965857E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.749999999E-01	1.200000000E+00	-8.500000051E-02	3.000000000E+00
-3.000000005E-01	8.999999999E+00	-1.999999998E-01	1.100000000E+01
-1.249999581E-01	3.999999994E+00	-9.999998924E-02	4.049999972E+00

CHARACTERISTIC VALUE DIFFERENCES

-1.450000000E-10	-2.600000000E-10	5.126000000E-10	-2.900000000E-10
4.600000000E-10	5.400000000E-10	-2.090000000E-10	2.000000000E-10
-4.187000000E-08	6.210000000E-09	-1.076100000E-08	2.814000000E-08

COMPUTED COEFFICIENTS

b_k			
-1.736481758E-01	9.848077525E-01	3.428281472E-01	9.396926254E-01
-7.660444437E-01	-6.427876135E-01	-9.396926203E-01	3.428281412E-01
9.396924228E-01	-3.428214179E-01	8.668255942E-01	5.000012727E-01

COEFFICIENT DIFFERENCES

-2.240000000E-09	5.070000000E-10	-4.167000000E-09	-4.376000000E-09
6.830000000E-10	3.464000000E-09	-7.230000000E-10	1.769000000E-09
1.981960000E-07	1.274930000E-06	-1.982200000E-07	-1.272664000E-06

TABLE 9C
COMPUTED DEFINING PARAMETERS FROM TRUNCATED FUNCTION VALUES

NUMBER OF ITERATIONS= 16

RADIUS OF DIFFERENCES 9.633412042E-08

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1.750272523E-01	1.200112477E+00	-8.506619756E-02	3.000007384E+00
-2.987580140E-01	9.000772484E+00	-2.000502121E-01	1.099998201E+01
-1.317445485E-01	3.999254730E+00	-1.029261047E-01	4.056833040E+00

CHARACTERISTIC VALUE DIFFERENCES

2.725232900E-05	-1.124774200E-04	6.619755680E-05	-7.383640000E-06
-1.241986024E-03	-7.724841500E-04	5.021208900E-05	1.798520000E-05
6.744548536E-03	7.452700300E-04	2.926104651E-03	-6.833039550E-03

COMPUTED COEFFICIENTS

b_k			
-1.735821396E-01	9.839918250E-01	3.423672743E-01	9.398274240E-01
-7.626946042E-01	-6.380793617E-01	-9.378672626E-01	3.424253723E-01
9.374991959E-01	-1.078903487E-01	8.635735663E-01	2.669755431E-01

COEFFICIENT DIFFERENCES

-6.603837900E-05	8.159280490E-04	-3.471312040E-04	6.651969940E-04
-3.349838847E-03	-4.748248274E-03	-1.825358396E-03	-4.052292720E-04
2.193425143E-03	-2.341297943E-01	2.451837710E-03	2.338244569E-01

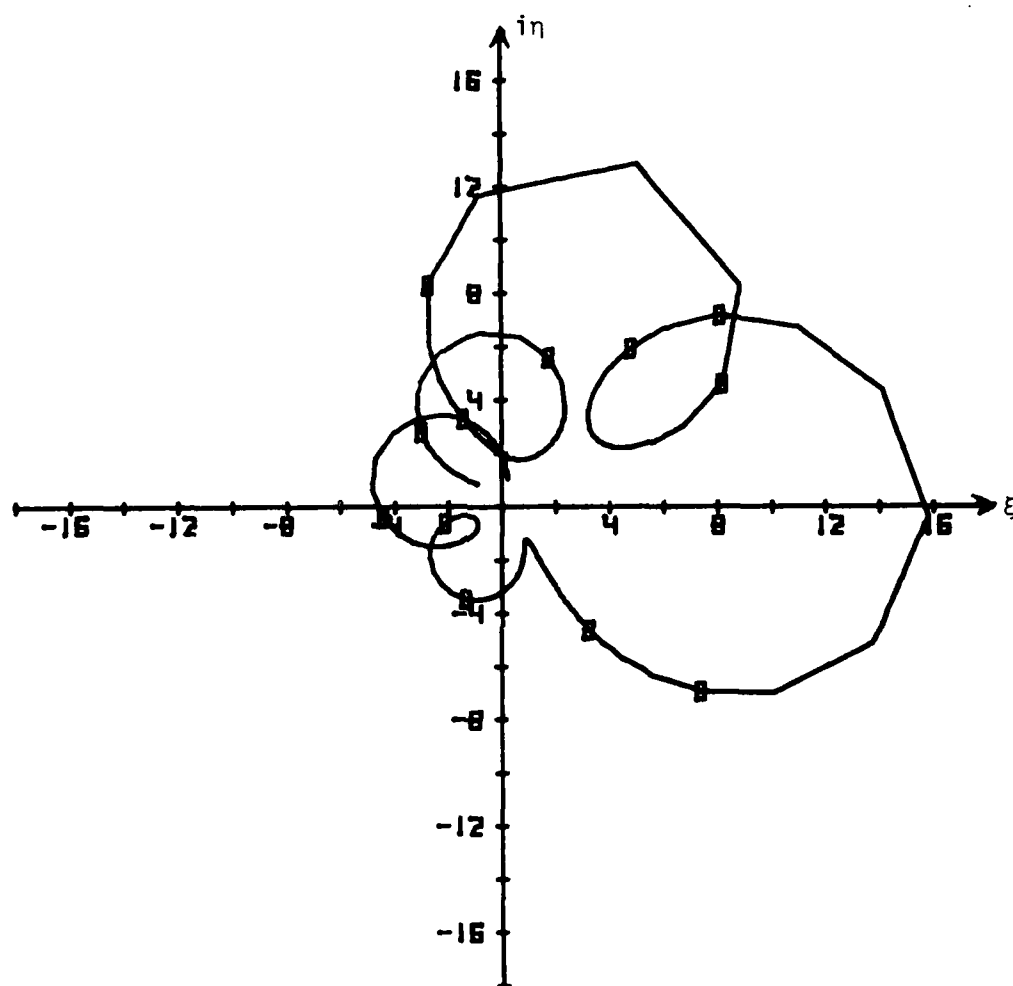


Figure 16a. Graph of $w(\omega)$, $\lambda_6 = -0.1 + 4.05i$

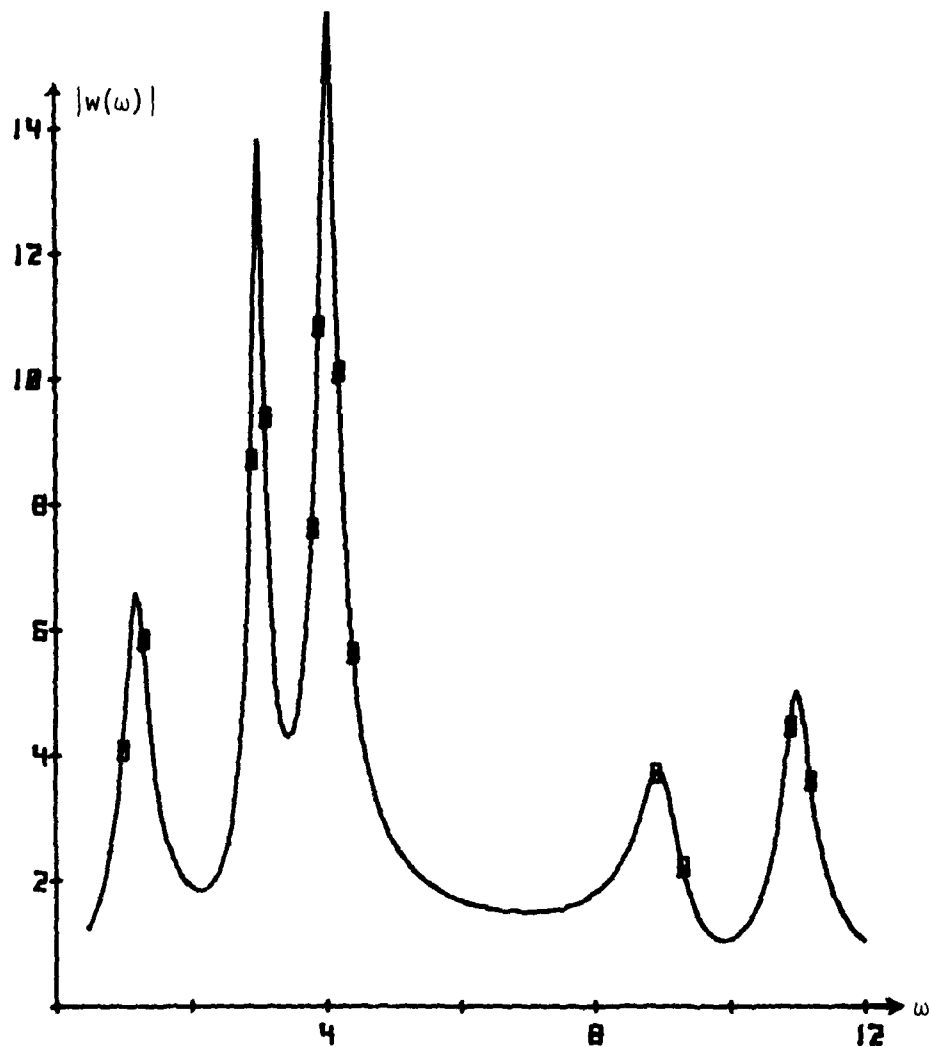


Figure 16b. Graph of $|w(\omega)|$, $\lambda_6 = -0.1 + 4.05i$

TABLE 9D

FUNCTION VALUES DETERMINED USING THE COMPUTED
CHARACTERISTIC VALUES AND COEFFICIENTS FROM TABLE 9C

COMPUTED CHARACTERISTIC VALUES

λ_k			
-1. 750272523E-01	1. 200112477E+00	-8. 506619756E-02	3. 000007304E+00
-2. 987500140E-01	9. 000772404E+00	-2. 000502121E-01	1. 099998201E+01
-1. 317445485E-01	3. 999254730E+00	-1. 029261047E-01	4. 056833040E+00

COMPUTED COEFFICIENTS

b_k			
-1. 735821396E-01	9. 839918250E-01	3. 423672743E-01	9. 390274240E-01
-7. 626946042E-01	-6. 300392617E-01	-9. 378672626E-01	3. 424253723E-01
9. 374991959E-01	-1. 078903487E-01	8. 635735663E-01	2. 669755431E-01

FREQUENCIES

ω_p	ω_{p+6}
1. 000000000E+00	1. 300000000E+00
2. 900000000E+00	3. 100000000E+00
8. 900000000E+00	9. 300000000E+00
1. 090000000E+01	1. 120000000E+01
3. 800000000E+00	3. 900000000E+00
4. 200000000E+00	4. 400000000E+00

FUNCTION VALUES BASED ON COMPUTED COEFFICIENTS
AND COMPUTED CHARACTERISTIC VALUES

-4. 339999982E+00	2. 289999980E+00
-3. 499999882E+00	7. 969999988E+00
-1. 689999997E+00	-3. 579999996E+00
-4. 720000006E+00	-5. 000000053E-01
4. 119999996E+00	5. 660000002E+00
6. 739999997E+00	-7. 179999997E+00
5. 400000166E-01	5. 129999983E+00
7. 410000130E+00	4. 309999976E+00
-2. 459999993E+00	-7. 399999938E-01
-1. 720000005E+00	3. 179999994E+00
7. 440000020E+00	6. 919999722E+00
2. 619999997E+00	-4. 849999999E+00

TRUE FUNCTION VALUES

-4. 342356007E+00	2. 293566793E+00
-3. 505942206E+00	7. 974718340E+00
-1. 699610241E+00	-3. 585619009E+00
-4. 729593563E+00	-5. 058363316E-01
4. 127276863E+00	5. 664619142E+00
6. 741516852E+00	-7. 183185670E+00
5. 450584266E-01	5. 132459956E+00
7. 419104404E+00	4. 315758260E+00
-2. 465389712E+00	-7. 468317069E-01
-1. 727659556E+00	3. 184497340E+00
7. 448711571E+00	6. 926368204E+00
2. 628041209E+00	-4. 859070040E+00

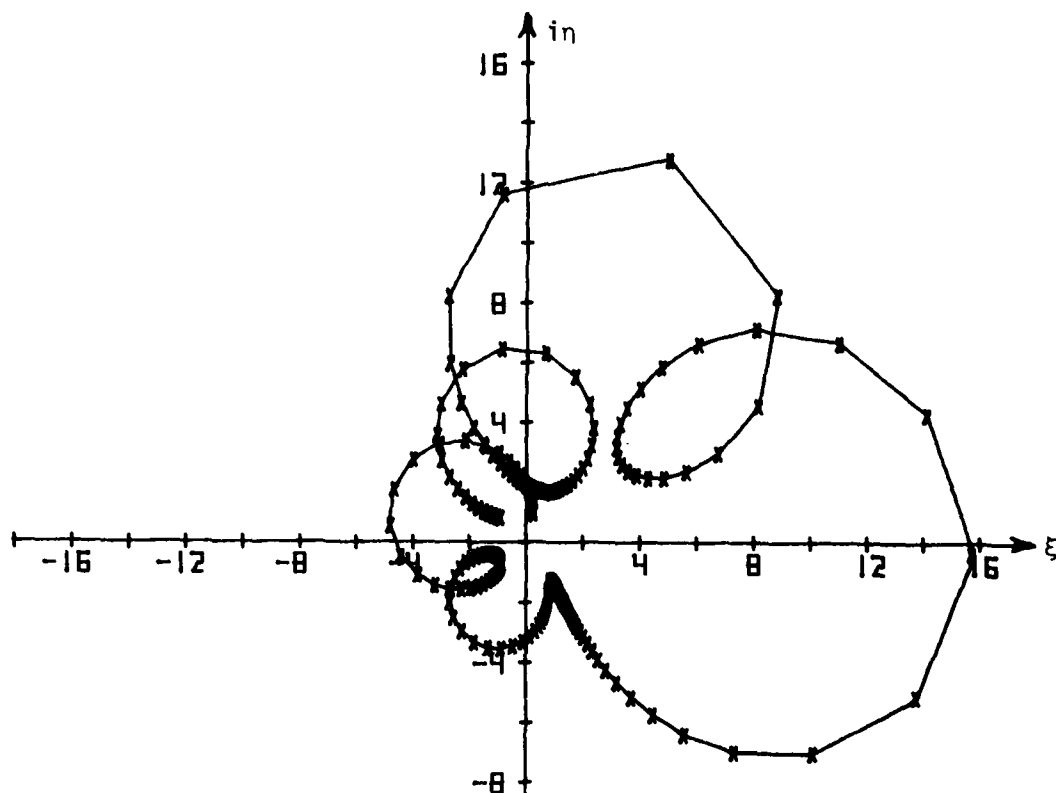


Figure 17. Comparison of Values of $w(\omega)$ for $w(\omega)$ Computed Using Exact Characteristic Values and Coefficients and $w(\omega)$ Computed Using the Characteristic Values and Coefficients From TABLE 9C

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